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Journal of Approximation Theory 125 (2003) 198–237

Journal of Approximation Theory

http://www.elsevier.com/locate/jat

Strong asymptotics of the recurrence coefficients of orthogonal polynomials associated to the generalized Jacobi weight

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Received 29 November 2002; accepted in revised form 3 November 2003

Communicated by Guillermo López Lagomasino

Abstract

We study asymptotics of the recurrence coefficients of orthogonal polynomials associated to the generalized Jacobi weight, which is a weight function with a finite number of algebraic singularities on $[-1, 1]$. The recurrence coefficients can be written in terms of the solution of the corresponding Riemann–Hilbert (RH) problem for orthogonal polynomials. Using the steepest descent method of Deift and Zhou, we analyze the RH problem, and obtain complete asymptotic expansions of the recurrence coefficients. We will determine explicitly the order $1/n$ terms in the expansions. A critical step in the analysis of the RH problem will be the local analysis around the algebraic singularities, for which we use Bessel functions of appropriate order. In addition, the RH approach gives us also strong asymptotics of the orthogonal polynomials near the algebraic singularities in terms of Bessel functions.

Keywords: Riemann–Hilbert problems; Steepest descent method; Generalized Jacobi weight; Bessel functions

1. Introduction

We consider the generalized Jacobi weight

$$w(x) = (1-x)^{\alpha}(1+x)^{\beta}h(x)\prod_{\nu=1}^{p}|x-x_\nu|^{2\lambda_\nu}, \quad \text{for } x\in(-1,1), \tag{1.1}$$

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0021-9045/$ - see front matter © 2003 Published by Elsevier Inc.
doi:10.1016/j.jat.2003.11.005

where p is a fixed number, with

$$-1 < x_1 < x_2 < \cdots < x_p < 1, \quad 2\lambda_\nu > -1, \lambda_\nu \neq 0, \quad \alpha, \beta > -1,$$

and with h real analytic and strictly positive on $[-1, 1]$. The points $x_1, \ldots, x_p$ are called the algebraic singularities of the weight. Throughout the paper we use $x_0 = -1$ and $x_{p+1} = 1$, for notational convenience. All the moments of w exist so that we have a sequence of orthogonal polynomials. Denote the nth degree orthonormal polynomial with respect to the generalized Jacobi weight by $p_n(z) = \gamma_n z^n + \cdots$, where $\gamma_n > 0$. These orthonormal polynomials satisfy a three term recurrence relation

$$xp_n(x) = a_{n+1}p_{n+1}(x) + b_n p_n(x) + a_n p_{n-1}(x),$$

and we will investigate the asymptotic behavior of the recurrence coefficients a_n and b_n as $n \to \infty$. The generalized Jacobi weight has been studied before from other points of view in [2,10,20,21] among others.

For the pure Jacobi weight $(1-x)^\alpha(1+x)^\beta$ exact expressions are known for the associated recurrence coefficients a_n and b_n, see [4,19]. The asymptotic behavior is given by

$$a_n = \frac{1}{2} + O(1/n^2), \quad b_n = O(1/n^2), \quad \text{as } n \to \infty.$$

In a previous paper with Kuijlaars et al. [16], we considered the modified Jacobi weight $(1-x)^\alpha(1+x)^\beta h(x)$. There, we were able to obtain complete asymptotic expansions of the associated recurrence coefficients in powers of $1/n$. It turned out that, as for the pure Jacobi weight, the order $1/n$ terms in the expansions vanished. The asymptotic behavior of the recurrence coefficients of orthogonal polynomials associated to the generalized Jacobi weight (1.1) has been studied before by Golinskii [13]. He has proven that

$$a_n = \frac{1}{2} + O(1/n), \quad b_n = O(1/n), \quad \text{as } n \to \infty. \tag{1.2}$$

In this paper we will give stronger asymptotics. We will prove that the $O(1/n)$ terms in (1.2) can be developed into complete asymptotic expansions in powers of $1/n$. Here, in contrast with the (modified) Jacobi weight, the order $1/n$ terms in the expansions will not vanish and we will determine an explicit expression for them.

Our approach is based on the characterization of orthogonal polynomials via a Riemann–Hilbert problem, due to Fokas et al. [11], and on an application of the steepest descent method for Riemann–Hilbert problems of Deift and Zhou [9]. We have already applied this technique to the modified Jacobi weight [16], and in our case, the general scheme is the same. The main difference lies in the fact that we now have to do a local analysis around the algebraic singularities as well (not just only around the endpoints ± 1), which will be done with (modified) Bessel functions of appropriate order. In the present paper, we will emphasize the construction of the local parametrix near the algebraic singularities, which is new. It will turn out that the order $1/n$ terms in the expansions of the recurrence coefficients come from this parametrix. The RH approach has been applied before to orthogonal polynomials, see [3,6–8,14–16]. Our result is the following.

Theorem 1.1. *The recurrence coefficients* a_n *and* b_n *of orthogonal polynomials associated to the generalized Jacobi weight* (1.1) *have a complete asymptotic expansion of the form*

$$a_n \sim \frac{1}{2} + \sum_{k=1}^{\infty} \frac{A_k(n)}{n^k}, \quad b_n \sim \sum_{k=1}^{\infty} \frac{B_k(n)}{n^k}, \tag{1.3}$$

as $n \to \infty$. *The coefficients* $A_k(n)$ *and* $B_k(n)$ *are explicitly computable for every* k, *and the coefficients with the* $1/n$ *term in the expansions are given by*

$$A_1(n) = -\frac{1}{2}\sum_{v=1}^{p} \lambda_v \sqrt{1-x_v^2}\cos(2n\arccos x_v - \Phi_v), \tag{1.4}$$

$$B_1(n) = -\sum_{v=1}^{p} \lambda_v \sqrt{1-x_v^2}\cos((2n+1)\arccos x_v - \Phi_v), \tag{1.5}$$

where

$$\begin{aligned}\Phi_v = {} & \left(\alpha + \lambda_v + \sum_{k=v+1}^{p} 2\lambda_k\right)\pi - \left(\alpha + \beta + \sum_{k=1}^{p} 2\lambda_k\right)\arccos x_v \\ & - \frac{\sqrt{1-x_v^2}}{\pi} \fint_{-1}^{1} \frac{\log h(t)}{\sqrt{1-t^2}} \frac{dt}{t - x_v}.\end{aligned} \tag{1.6}$$

The integral in (1.6) *is a Cauchy principal value integral.*

This theorem shows that $n(2a_n - 1)$ and nb_n are oscillatory and asymptotically behave like a superposition of p wave functions $A_v \cos(\omega_v n - \phi_v)$ with amplitudes $A_v = -\lambda_v\sqrt{1-x_v^2}$, frequencies $\omega_v = 2\arccos x_v$, and phase shifts ϕ_v which are different for $n(2a_n - 1)$ and nb_n. The amplitude A_v depends on the location and the strength of the singularity x_v, while the frequency ω_v depends only on the location of x_v. The strengths of the other singularities has influence on the phase shift ϕ_v. This discussion shows that the $O(1/n)$ behavior of the recurrence coefficients is intimately related to the behavior of our weight near the singularities. Note that if we have no singularities (i.e. $\lambda_1 = \cdots = \lambda_p = 0$) all the amplitudes in the wave functions vanish. This implies that the order $1/n$ terms in the expansions of the recurrence coefficients vanish, which is in agreement with the case of the modified Jacobi weight [16].

Remark 1.2. We have restricted ourselves to determine only the order $1/n$ terms in the expansions of the recurrence coefficients. It is possible to determine the higher-order terms in the same way if we work hard enough, but the calculations will be a mess.

We will now compare our result with a conjecture of Magnus [17] about the asymptotic behavior of the recurrence coefficients of orthogonal polynomials associated to the weight

$$w(x) = \begin{cases} B(1-x)^{\alpha}(1+x)^{\beta}(x_1-x)^{2\lambda}, & \text{for } x\in(-1,x_1), \\ A(1-x)^{\alpha}(1+x)^{\beta}(x-x_1)^{2\lambda}, & \text{for } x\in(x_1,1), \end{cases} \tag{1.7}$$

where A and B are positive constants, with $-1<x_1<1$, where $\alpha,\beta>-1$ and where $2\lambda>-1$. This weight allows a jump at x_1, and is of form (1.1) only if $A=B$. The conjecture is the following.

Conjecture of Magnus [17]. The recurrence coefficients of orthogonal polynomials associated to the weight (1.7) satisfy

$$a_n = \frac{1}{2} - \frac{M}{n}\cos(2n \arccos x_1 - 4\mu\log(4n\sin\arccos x_1) - \Phi) + o(1/n), \tag{1.8}$$

$$b_n = -\frac{2M}{n}\cos((2n+1)\arccos x_1 - 4\mu\log(4n\sin\arccos x_1) - \Phi) + o(1/n), \tag{1.9}$$

as $n\to\infty$. Here

$$\mu = \frac{1}{2\pi}\log(B/A), \quad M = \frac{1}{2}\sqrt{\lambda^2+\mu^2}\sqrt{1-x_1^2}, \tag{1.10}$$

$$\Phi = (\alpha+\lambda)\pi - (\alpha+\beta+2\lambda)\arccos x_1 - 2\arg\Gamma(\lambda+i\mu) - \arg(\lambda+i\mu). \tag{1.11}$$

We want to show that, as a consequence of Theorem 1.1, the conjecture is true for the case $A=B$. To this end, we need to reformulate the conjecture for this case. If $A=B$ we have by (1.10) and (1.11) that $\mu=0$, $M=(|\lambda|/2)\sqrt{1-x_1^2}$, and

$$\Phi = \begin{cases} (\alpha+\lambda)\pi - (\alpha+\beta+2\lambda)\arccos x_1, & \text{if } \lambda>0, \\ (\alpha+\lambda)\pi - (\alpha+\beta+2\lambda)\arccos x_1 - 3\pi, & \text{if } \lambda<0. \end{cases}$$

Inserting this into (1.8) and (1.9) the conjecture becomes.

Conjecture of Magnus for the case $A=B$. The recurrence coefficients of orthogonal polynomials associated to the weight $A(1-x)^{\alpha}(1+x)^{\beta}|x-x_1|^{2\lambda}$ satisfy

$$a_n = \frac{1}{2} - \frac{\lambda}{2n}\sqrt{1-x_1^2}\cos(2n\arccos x_1 - \hat{\Phi}) + o(1/n),$$

$$b_n = -\frac{\lambda}{n}\sqrt{1-x_1^2}\cos((2n+1)\arccos x_1 - \hat{\Phi}) + o(1/n),$$

as $n\to\infty$, where $\hat{\Phi} = (\alpha+\lambda)\pi - (\alpha+\beta+2\lambda)\arccos x_1$.

If we apply Theorem 1.1 for the case $p=1$ and $h=A$, and using the fact that, see [12],

$$\fint_{-1}^{1} \frac{1}{\sqrt{1-t^2}} \frac{dt}{t-x_1} = 0,$$

we see that the conjecture is true for the case $A=B$. We even have more since we were able to obtain complete asymptotic expansions of the recurrence coefficients, allow the analytic factor h, and allow more singularities. The full conjecture (i.e. $A \neq B$) remains open.

We will also compare our result with Nevai's result [18] for an even positive weight $\rho(x)|x|^{2\lambda}$ on $[-1,1]$ with ρ and ρ' continuous and with $2\lambda > -1$. Nevai showed in [18] that the recurrence coefficient a_n of orthogonal polynomials associated to this weight satisfies,

$$a_n = \frac{1}{2} + (-1)^{n+1} \frac{\lambda}{2n} + o(1/n), \quad \text{as } n \to \infty. \tag{1.12}$$

We apply Theorem 1.1 to the weight $w(x) = (1-x^2)^{\alpha} h(x) |x|^{2\lambda_1}$, with h even. So, w is of the form of Nevai's weight. Using the fact that h is even and $x_1 = 0$, we have

$$\fint_{-1}^{1} \frac{\log h(t)}{\sqrt{1-t^2}} \frac{dt}{t-x_1} = 0$$

since the integrand is an odd function. Since $p=1$, $\alpha=\beta$, and $\arccos x_1 = \pi/2$ this implies by (1.6) that the phase constant Φ_1 vanishes. From (1.3) and (1.4) we then have

$$a_n = \frac{1}{2} + (-1)^{n+1} \frac{\lambda_1}{2n} + O(1/n^2), \quad \text{as } n \to \infty.$$

This is in agreement with Nevai's result, see (1.12). The error is stronger since we have $O(1/n^2)$ instead of $o(1/n)$. However, here we are dealing with an even weight of the form $\rho(x)|x|^{2\lambda}$ with ρ analytic, and Nevai's weights also include cases where ρ is non-analytic. Nevai also showed that the error is $O(1/n^2)$ if ρ is constant. Note, since the phase constant Φ_1 vanishes, that by (1.5) the order $1/n$ term in the expansion of b_n vanishes. This is in agreement with the fact that $b_n = 0$ for an even weight.

From our analysis we are also able to determine strong asymptotics of the orthonormal polynomials p_n. In particular we are interested in the asymptotics near the singularities. The result is the following.

Theorem 1.3. *Fix $v \in \{1,\dots,p\}$. There exists $\delta > 0$ so that for $x \in (x_v, x_v + \delta)$*

$$\begin{aligned} p_n(x) = {} & \frac{(n\theta_v)^{1/2}}{\sqrt{w(x)}(1-x^2)^{1/4}} \\ & \times [(1+O(1/n))(\cos \zeta_1(x) J_{\lambda_v - \frac{1}{2}}(n\theta_v) + \sin \zeta_1(x) J_{\lambda_v + \frac{1}{2}}(n\theta_v)) \\ & + O(1/n)(\cos \zeta_2(x) J_{\lambda_v - \frac{1}{2}}(n\theta_v) + \sin \zeta_2(x) J_{\lambda_v + \frac{1}{2}}(n\theta_v))], \end{aligned} \tag{1.13}$$

as $n \to \infty$, *with* $\theta_\nu = \arccos x_\nu - \arccos x$, *and with* J_ν *the usual J-Bessel function of order* ν. *The error terms hold uniformly for* $x \in (x_\nu, x_\nu + \delta)$ *and have a full asymptotic expansion in powers of* $1/n$, *which can be calculated explicitly. In* (1.13),

$$\zeta_{1,2}(x) = \pm \frac{1}{2} \arccos x + \psi_\nu(x) - \frac{\lambda_\nu \pi}{2} + n \arccos x_\nu - \frac{\pi}{4}, \tag{1.14}$$

where the $+$ *holds for* ζ_1, *the* $-$ *for* ζ_2, *and where* ψ_ν *is given by*

$$\psi_\nu(x) = -\frac{1}{2}\left[\left(\alpha + \sum_{k=\nu+1}^{p} 2\lambda_k \right) \pi - \left(\alpha + \beta + \sum_{k=1}^{p} 2\lambda_k \right) \arccos x \right. \\ \left. - \frac{\sqrt{1-x^2}}{\pi} \fint_{-1}^{1} \frac{\log h(t)}{\sqrt{1-t^2}} \frac{dt}{t-x} \right]. \tag{1.15}$$

The integral in (1.15) *is a Cauchy principal value integral.*

The present paper is organized as follows. In Section 2 we formulate the theory of orthogonal polynomials as a RH problem for Y. In Section 3, we do the asymptotic analysis of this RH problem. There, we want to obtain, via a series of transformations $Y \mapsto T \mapsto S \mapsto R$, a RH problem for R with a jump matrix close to the identity matrix. Then, R is also close to the identity matrix. For the last transformation $S \mapsto R$ we have to do a local analysis near the endpoints and near the algebraic singularities. The local analysis near the endpoints has already been done in [16], but near the algebraic singularities it is new and will be done in Section 4. In Section 5 we determine a complete asymptotic expansion of the jump matrix for R. As a result, we obtain a complete asymptotic expansion of R, which will be used to prove Theorem 1.1. In the last section we determine the asymptotics of the orthonormal polynomials near the algebraic singularities.

2. RH problem for Y

In this section we will characterize the orthogonal polynomials via a 2×2 matrix valued RH problem. This characterization is due to Fokas et al. [11]. We will also write down the recurrence coefficients a_n and b_n in terms of the solution of this RH problem.

We seek a 2×2 matrix valued function $Y(z) = Y(z; n, w)$ that satisfies the following RH problem.

RH problem for $\boldsymbol{Y}$**.**

(a) $Y(z)$ is analytic for $z \in \mathbb{C} \setminus [-1, 1]$.
(b) Y possesses continuous boundary values for $x \in (-1, 1) \setminus \{x_1, \ldots, x_p\}$ denoted by $Y_+(x)$ and $Y_-(x)$, where $Y_+(x)$ and $Y_-(x)$ denote the limiting values of $Y(z')$ as

z' approaches x from above and below, respectively, and

$$Y_+(x) = Y_-(x)\begin{pmatrix} 1 & w(x) \\ 0 & 1 \end{pmatrix}, \quad \text{for } x \in (-1,1) \setminus \{x_1, \ldots, x_p\}. \tag{2.1}$$

(c) $Y(z)$ has the following asymptotic behavior at infinity:

$$Y(z) = (I + O(1/z))\begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}, \quad \text{as } z \to \infty. \tag{2.2}$$

(d) $Y(z)$ has the following behavior near $z = 1$:

$$Y(z) = \begin{cases} O\begin{pmatrix} 1 & |z-1|^\alpha \\ 1 & |z-1|^\alpha \end{pmatrix}, & \text{if } \alpha < 0, \\ O\begin{pmatrix} 1 & \log|z-1| \\ 1 & \log|z-1| \end{pmatrix}, & \text{if } \alpha = 0, \\ O\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, & \text{if } \alpha > 0, \end{cases} \tag{2.3}$$

as $z \to 1$, $z \in \mathbb{C} \setminus [-1,1]$.

(e) $Y(z)$ has the following behavior near $z = -1$:

$$Y(z) = \begin{cases} O\begin{pmatrix} 1 & |z+1|^\beta \\ 1 & |z+1|^\beta \end{pmatrix}, & \text{if } \beta < 0, \\ O\begin{pmatrix} 1 & \log|z+1| \\ 1 & \log|z+1| \end{pmatrix}, & \text{if } \beta = 0, \\ O\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, & \text{if } \beta > 0, \end{cases} \tag{2.4}$$

as $z \to -1$, $z \in \mathbb{C} \setminus [-1,1]$.

(f) $Y(z)$ has the following behavior near $z = x_v$, for every $v = 1, \ldots, p$:

$$Y(z) = \begin{cases} O\begin{pmatrix} 1 & |z-x_v|^{2\lambda_v} \\ 1 & |z-x_v|^{2\lambda_v} \end{pmatrix}, & \text{if } \lambda_v < 0, \\ O\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, & \text{if } \lambda_v > 0, \end{cases} \tag{2.5}$$

as $z \to x_v$, $z \in \mathbb{C} \setminus [-1,1]$.

Remark 2.1. The O-terms in (2.3)–(2.5) are to be taken entrywise. So for example $Y(z) = O\begin{pmatrix} 1 & |z-1|^\alpha \\ 1 & |z-1|^\alpha \end{pmatrix}$ means that $Y_{11}(z) = O(1)$, $Y_{12}(z) = O(|z-1|^\alpha)$, etc.

If we take care of the algebraic singularities x_ν of the generalized Jacobi weight in the same way as of the endpoints ± 1 in [16, Section 2] we obtain the following theorem.

Theorem 2.2. *The RH problem for* Y *has a unique solution* $Y(z) = Y(z; n, w)$ *given by*,

$$Y(z) = \begin{pmatrix} \pi_n(z) & \dfrac{1}{2\pi i}\displaystyle\int_{-1}^{1} \frac{\pi_n(x)w(x)}{x-z}\,dx \\ -2\pi i\gamma_{n-1}^2\pi_{n-1}(z) & -\gamma_{n-1}^2 \displaystyle\int_{-1}^{1} \frac{\pi_{n-1}(x)w(x)}{x-z}\,dx \end{pmatrix}, \tag{2.6}$$

where π_n *is the monic polynomial of degree* n *orthogonal with respect to the weight* w *and with* γ_n *the leading coefficient of the orthonormal polynomial* p_n.

The recurrence coefficients a_n and b_n can be written in terms of Y, see [5,7,11,16]. It is known [5] that

$$a_n^2 = \lim_{z\to\infty} z^2 Y_{12}(z; n, w) Y_{21}(z; n, w), \tag{2.7}$$

$$b_n = \lim_{z\to\infty} (z - Y_{11}(z; n+1, w) Y_{22}(z; n, w)). \tag{2.8}$$

So, in order to determine the asymptotics of the recurrence coefficients, we need to do an asymptotic analysis of the RH problem for Y.

3. Asymptotic analysis of the RH problem for Y

In this section we will do the asymptotic analysis of the RH problem for Y. The idea is to obtain, via a series of transformations

$$Y \mapsto T \mapsto S \mapsto R,$$

a RH problem for R which is normalized at infinity (i.e. $R(z) \to I$ as $z \to \infty$) and whose jump matrix is close to the identity matrix. As a result, the solution of the RH problem for R is also close to the identity matrix, cf. [5,7].

As mentioned in the introduction, we point out that the asymptotic analysis is analogous as in the case of the modified Jacobi weight, see [16]. The main differences, which come from the algebraic singularities, are:

- In every step we have to take care of the growth condition near the algebraic singularities, which was included in the RH problem for Y to control the behavior near these points.
- In the second transformation $T \to S$ the lens will be opened going through the algebraic singularities.
- We have to do a local analysis around the algebraic singularities, not just only around the endpoints. This is a new and critical step in the analysis of the RH problem for Y, and is the most important difference with the case of the modified

Jacobi weight. To emphasize this, the construction of the parametrix near the algebraic singularities will be done in a separate section.

3.1. First transformation $Y \mapsto T$

We will first transform the RH problem for Y into a RH problem for T whose solution is bounded at infinity, and whose jump matrix has oscillatory diagonal entries. Let $\varphi(z) = z + (z^2 - 1)^{1/2}$ be the conformal mapping that maps $\mathbb{C}\setminus[-1,1]$ onto the exterior of the unit circle, and define

$$T(z) = 2^{n\sigma_3} Y(z) \varphi(z)^{-n\sigma_3}, \quad \text{for } z \in \mathbb{C}\setminus[-1,1], \tag{3.1}$$

where $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is the Pauli matrix. With Pauli's notation x^{σ_3} we mean

$$x^{\sigma_3} = \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix},$$

for some scalar x. Then, T is the unique solution of the following equivalent RH problem, cf. [16, Section 3].

RH problem for T.

(a) $T(z)$ is analytic for $z \in \mathbb{C}\setminus[-1,1]$.
(b) $T(z)$ satisfies the following jump relation on $(-1,1)\setminus\{x_1,\dots,x_p\}$:

$$T_+(x) = T_-(x) \begin{pmatrix} \varphi_+(x)^{-2n} & w(x) \\ 0 & \varphi_-(x)^{-2n} \end{pmatrix}, \quad \text{for } x \in (-1,1)\setminus\{x_1,\dots,x_p\}. \tag{3.2}$$

(c) $T(z)$ has the following behavior at infinity:

$$T(z) = I + O(1/z), \quad \text{as } z \to \infty. \tag{3.3}$$

(d) $T(z)$ has the same behavior as $Y(z)$ as $z \to 1$, given by (2.3).
(e) $T(z)$ has the same behavior as $Y(z)$ as $z \to -1$, given by (2.4).
(f) $T(z)$ has the same behavior as $Y(z)$ as $z \to x_v$, given by (2.5), for every $v = 1, \dots, p$.

Remark 3.1. Condition (c) states that the RH problem for T is normalized at infinity. Since $|\varphi_\pm(x)| = 1$ for $x \in (-1,1)$ we have by (3.2) oscillatory diagonal entries in the jump matrix for T.

3.2. Second transformation $T \to S$

We use the steepest descent method for RH problems of Deift and Zhou [9] to remove the oscillatory behavior in (3.2). See [5,8] for an introduction. The idea is to deform the contour so that the oscillatory diagonal entries in the jump matrix for T are transformed into exponentially decaying off-diagonal entries. We then arrive at an equivalent RH problem for S on a lens shaped contour, with jumps that converge to the identity matrix on the lips of the lens, as $n \to \infty$. This step is referred to as the *opening of the lens*.

Since h is real analytic and strictly positive on $[-1, 1]$, there is a neighborhood U of $[-1, 1]$ so that h has an analytic continuation to U, and so that the real part of h is strictly positive on U. Hence, the factor $(1-x)^{\alpha}(1+x)^{\beta}h(x)$ has a non-vanishing analytic continuation to $z \in U \setminus ((-\infty, -1] \cup [1, \infty))$, given by

$$(1-z)^{\alpha}(1+z)^{\beta}h(z),$$

with principal branches of powers.

To continuate the factor $|x - x_{\nu}|^{2\lambda_{\nu}}$ analytically, where $\nu \in \{1, \dots, p\}$, we divide the complex plane into two regions, which we denote by $K^{\mathrm{l}}_{x_{\nu}}$ and $K^{\mathrm{r}}_{x_{\nu}}$, separated by a contour $\Gamma_{x_{\nu}}$ going through x_{ν}, see Fig. 1. Here, $K^{\mathrm{l}}_{x_{\nu}}$ and $K^{\mathrm{r}}_{x_{\nu}}$ are the sets of all points on the left, respectively right, of $\Gamma_{x_{\nu}}$. We choose the contour $\Gamma_{x_{\nu}}$ so that the images of $\Gamma_{x_{\nu}} \cap \mathbb{C}_+$ and $\Gamma_{x_{\nu}} \cap \mathbb{C}_-$, under the mapping φ, are the straight rays, restricted to the exterior of the unit circle, with arguments $\arccos x_{\nu}$ and $-\arccos x_{\nu}$, respectively. Here, $\mathbb{C}_+$ is used to denote the upper half-plane $\{z \mid \operatorname{Im} z > 0\}$, and $\mathbb{C}_-$ to denote the lower half-plane $\{z \mid \operatorname{Im} z < 0\}$. It turns out that $\Gamma_{x_{\nu}}$ is a hyperbola and goes vertically through x_{ν}. We have made an exact plot of $\Gamma_{x_{\nu}}$ for the case $x_{\nu} = \frac{1}{2}$, see Fig. 1. For $\nu = 1, \dots, p$, the factor $|x - x_{\nu}|^{2\lambda_{\nu}}$ has an analytic continuation to $z \in \mathbb{C} \setminus \Gamma_{x_{\nu}}$, given by

$$\begin{cases} (x_{\nu} - z)^{2\lambda_{\nu}}, & \text{for } z \in K^{\mathrm{l}}_{x_{\nu}}, \\ (z - x_{\nu})^{2\lambda_{\nu}}, & \text{for } z \in K^{\mathrm{r}}_{x_{\nu}}, \end{cases}$$

with again principal branches of powers.

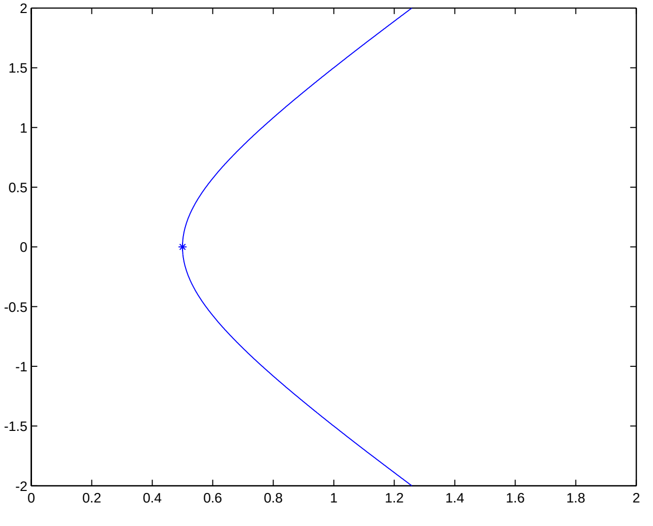

Fig. 1. The contour $\Gamma_{x_{\nu}}$ for the case $x_{\nu} = \frac{1}{2}$. The star marks x_{ν}, and $K^{\mathrm{l}}_{x_{\nu}}$ and $K^{\mathrm{r}}_{x_{\nu}}$ are the sets of all points on the left, respectively right, of $\Gamma_{x_{\nu}}$.

Remark 3.2. It seems a bit awkward to work with this choice of Γ_{x_v} instead of with the vertical line going through x_v, but in Section 4.2 this will become clear.

As a result, the generalized Jacobi weight w, given by (1.1), has a non-vanishing analytic continuation to $z \in U \backslash ((-\infty, -1] \cup [1, \infty) \cup [\bigcup_{v=1}^{p} \Gamma_{x_v}])$, also denoted by w, given by

$$w(z) = (1-z)^{\alpha}(1+z)^{\beta} h(z) \times \prod_{k=1}^{v} (z - x_k)^{2\lambda_k} \prod_{l=v+1}^{p} (x_l - z)^{2\lambda_l}, \quad \text{if } z \in K_{x_v}^{\mathrm{r}} \cap K_{x_{v+1}}^{\mathrm{l}}, \tag{3.4}$$

where $v = 0, \ldots, p$, and $K_{x_0}^{\mathrm{r}} = K_{x_{p+1}}^{\mathrm{l}} = \mathbb{C}$.

Remark 3.3. Note that only if $\lambda_v \in \mathbb{N}$ the analytic continuation of our weight is also analytic across the contour Γ_{x_v}.

The jump matrix (3.2) for T has the following factorization into a product of three matrices, based on the fact that $\varphi_+(x)\varphi_-(x) = 1$ for $x \in (-1, 1)$,

$$\begin{pmatrix} \varphi_+(x)^{-2n} & w(x) \\ 0 & \varphi_-(x)^{-2n} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ w(x)^{-1}\varphi_-(x)^{-2n} & 1 \end{pmatrix} \begin{pmatrix} 0 & w(x) \\ -w(x)^{-1} & 0 \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ w(x)^{-1}\varphi_+(x)^{-2n} & 1 \end{pmatrix}. \tag{3.5}$$

We note that $1/w$ does not have an analytic extension to a full neighborhood of $(-1, 1)$. Instead, it has an analytic continuation to a neighborhood of (x_v, x_{v+1}), for every $v = 0, \ldots, p$, where $x_0 = -1$ and $x_{p+1} = 1$. We thus transform the RH problem for T into a RH problem for S with jumps on the oriented contour Σ, shown in Fig. 2, that goes through the algebraic singularities x_v. The precise form of the lens Σ will be determined in Section 4.2. Of course it will be contained in U. We write

$$\Sigma^{\mathrm{o}} = \Sigma \backslash \{-1, x_1, \ldots, x_p, 1\}.$$

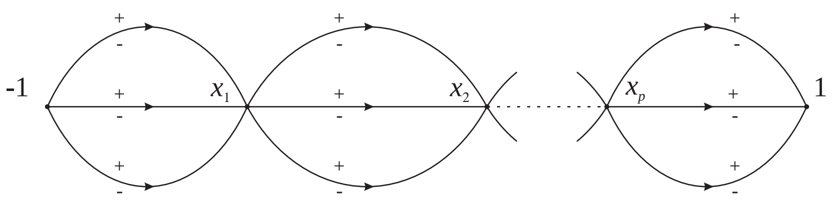

Fig. 2. The contour Σ.

Let us define, as in [16, Section 4],

$$S(z) = \begin{cases} T(z), & \text{for } z \text{ outside the lens,} \\ T(z)\begin{pmatrix} 1 & 0 \\ -w(z)^{-1}\varphi(z)^{-2n} & 1 \end{pmatrix}, & \text{for } z \text{ in the upper parts of the lens,} \\ T(z)\begin{pmatrix} 1 & 0 \\ w(z)^{-1}\varphi(z)^{-2n} & 1 \end{pmatrix}, & \text{for } z \text{ in the lower parts of the lens.} \end{cases} \tag{3.6}$$

Then, S is the unique solution of the following equivalent RH problem, cf. [16, Section 4].

RH problem for S.

(a) $S(z)$ is analytic for $z \in \mathbb{C} \setminus \Sigma$.
(b) $S(z)$ satisfies the following jump relations on Σ^{o},

$$S_+(z) = S_-(z)\begin{pmatrix} 1 & 0 \\ w(z)^{-1}\varphi(z)^{-2n} & 1 \end{pmatrix}, \quad \text{for } z \in \Sigma^{o} \cap (\mathbb{C}_+ \cup \mathbb{C}_-), \tag{3.7}$$

$$S_+(x) = S_-(x)\begin{pmatrix} 0 & w(x) \\ -w(x)^{-1} & 0 \end{pmatrix}, \quad \text{for } x \in \Sigma^{o} \cap (-1, 1). \tag{3.8}$$

(c) $S(z)$ has the following behavior at infinity:

$$S(z) = I + O(1/z), \quad \text{as } z \to \infty. \tag{3.9}$$

(d) For $\alpha < 0$, $S(z)$ has the following behavior as $z \to 1$:

$$S(z) = O\begin{pmatrix} 1 & |z-1|^{\alpha} \\ 1 & |z-1|^{\alpha} \end{pmatrix}, \quad \text{as } z \to 1, z \in \mathbb{C} \setminus \Sigma. \tag{3.10}$$

For $\alpha = 0$, $S(z)$ has the following behavior as $z \to 1$:

$$S(z) = O\begin{pmatrix} \log|z-1| & \log|z-1| \\ \log|z-1| & \log|z-1| \end{pmatrix}, \quad \text{as } z \to 1, z \in \mathbb{C} \setminus \Sigma. \tag{3.11}$$

For $\alpha > 0$, $S(z)$ has the following behavior as $z \to 1$:

$$S(z) = \begin{cases} O\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, & \text{as } z \to 1 \text{ from outside the lens,} \\ O\begin{pmatrix} |z-1|^{-\alpha} & 1 \\ |z-1|^{-\alpha} & 1 \end{pmatrix}, & \text{as } z \to 1 \text{ from inside the lens.} \end{cases} \tag{3.12}$$

(e) $S(z)$ has the same behavior near -1 if we replace in (3.10)–(3.12), α by β, $|z-1|$ by $|z+1|$ and take the limit $z \to -1$ instead of $z \to 1$.

(f) For $v = 1, \dots, p$, $S(z)$ has the following behavior as $z \to x_v$. For $\lambda_v < 0$ we have

$$S(z) = O\begin{pmatrix} 1 & |z - x_v|^{2\lambda_v} \\ 1 & |z - x_v|^{2\lambda_v} \end{pmatrix}, \quad \text{as } z \to x_v, z \in \mathbb{C} \setminus \Sigma. \tag{3.13}$$

For $\lambda_v > 0$ we have

$$S(z) = \begin{cases} O\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, & \text{as } z \to x_v \text{ from outside the lens,} \\ O\begin{pmatrix} |z - x_v|^{-2\lambda_v} & 1 \\ |z - x_v|^{-2\lambda_v} & 1 \end{pmatrix}, & \text{as } z \to x_v \text{ from inside the lens.} \end{cases} \tag{3.14}$$

Since $|\varphi(z)| > 1$ for $z \in \mathbb{C} \setminus [-1, 1]$ we see from (3.7) that the oscillatory terms on the diagonal entries in the jump matrix for T have been transformed into exponentially decaying off-diagonal entries in the jump matrix for S on the lips of the lens. So, the jump matrix for S converges exponentially fast to the identity matrix on the lips of the lens, as $n \to \infty$. Hence, we expect that the leading order asymptotics are determined by the solution of the following RH problem.

RH problem for N.

(a) $N(z)$ is analytic for $z \in \mathbb{C} \setminus [-1, 1]$.
(b) $N(z)$ satisfies the following jump relation on the interval $(-1, 1) \setminus \{x_1, \dots, x_p\}$:

$$N_+(x) = N_-(x) \begin{pmatrix} 0 & w(x) \\ -w(x)^{-1} & 0 \end{pmatrix}, \quad \text{for } x \in (-1, 1) \setminus \{x_1, \dots, x_p\}. \tag{3.15}$$

(c) $N(z)$ has the following behavior at infinity:

$$N(z) = I + O(1/z), \quad \text{as } z \to \infty. \tag{3.16}$$

The solution of the RH problem for N is referred to as the parametrix for the outside region and it has been solved in [16, Section 5] using the Szegő function associated with the generalized Jacobi weight w,

$$D(z) = \frac{(z-1)^{\alpha/2}(z+1)^{\beta/2} \prod_{v=1}^{p} (z - x_v)^{\lambda_v}}{\varphi(z)^{\left(\alpha + \beta + 2\sum_{v=1}^{p} \lambda_v\right)/2}} \times \exp\left(\frac{(z^2 - 1)^{1/2}}{2\pi} \int_{-1}^{1} \frac{\log h(x)}{\sqrt{1 - x^2}} \frac{dx}{z - x} \right). \tag{3.17}$$

The Szegő function $D(z)$ associated to w is analytic and non-zero for $z \in \mathbb{C} \setminus [-1, 1]$, satisfies the jump condition $D_+(x)D_-(x) = w(x)$ for $x \in (-1, 1) \setminus \{x_1, \dots, x_p\}$, and $D_\infty = \lim_{z \to \infty} D(z) \in (0, +\infty)$. The solution of the RH problem for N is then given by, see [16, Section 5],

$$N(z) = D_\infty^{\sigma_3} \begin{pmatrix} \dfrac{a(z)+a(z)^{-1}}{2} & \dfrac{a(z)-a(z)^{-1}}{2i} \\ \dfrac{a(z)-a(z)^{-1}}{-2i} & \dfrac{a(z)+a(z)^{-1}}{2} \end{pmatrix} D(z)^{-\sigma_3}, \tag{3.18}$$

where

$$a(z) = \frac{(z-1)^{1/4}}{(z+1)^{1/4}}. \tag{3.19}$$

For later use we have the following lemma.

Lemma 3.4. *For every* $v = 0, \dots, p$,

$$D_+(x) = \sqrt{w(x)}e^{-i\psi_v(x)}, \quad \text{for } x_v < x < x_{v+1}. \tag{3.20}$$

Here $x_0 = -1$, $x_{p+1} = 1$, *and* ψ_v *is given by* (1.15).

Proof. We rewrite expression (3.17) for the Szegő function as

$$D(z) = \frac{(z-1)^{\alpha/2}(z+1)^{\beta/2}\prod_{k=1}^{p}(z-x_k)^{\lambda_k}}{\varphi(z)^{\left(\alpha+\beta+2\sum_{k=1}^{p}\lambda_k\right)/2}} \exp(-i(z^2-1)^{1/2}\Phi(z)), \tag{3.21}$$

where

$$\Phi(z) = \frac{1}{2\pi i}\int_{-1}^{1} \frac{\log h(x)}{\sqrt{1-t^2}} \frac{dt}{t-z}.$$

Now, we determine $D_+(x)$ for $x \in (x_v, x_{v+1})$. So, we need to take the $+$ boundary values for all quantities in (3.21). Using the Sokhotskii–Plemelj formula [12, Section 4.2] we have

$$\Phi_+(x) = \frac{\log h(x)}{2\sqrt{1-x^2}} + \frac{1}{2\pi i} \fint_{-1}^{1} \frac{\log h(t)}{\sqrt{1-t^2}} \frac{dt}{t-x},$$

where the integral is a Cauchy principal value integral, so that, by (3.21) and the fact that $\varphi_+(x) = \exp(i \arccos x)$, the lemma is proved after an easy calculation. $\square$

Before we can to do the third transformation we have to be careful, since the jump matrices for S and N are not uniformly close to each other near the endpoints ± 1 and near the algebraic singularities x_v. Therefore, a local analysis near these points is necessary. Near the endpoints this has already been done in [16, Section 6].

We have constructed in [16, Section 6] a parametrix P_1 in the disk $U_{\delta,1}$ with radius $\delta>0$, sufficiently small, and center 1. This is a matrix valued function in $U_{\delta,1}$, that has the same jumps as S on Σ, that matches with N on the boundary $\partial U_{\delta,1}$ of $U_{\delta,1}$,

$$P_1(z)N^{-1}(z) = I + O(1/n), \quad \text{as } n\to\infty, \text{ uniformly for } z\in\partial U_{\delta,1}, \tag{3.22}$$

and that has the same behavior as $S(z)$ near $z=1$. The parametrix P_1 is given in [16, Section 6], and is constructed out of Bessel function of order α. We note that the scalar function W in [16, (6.27)], because of the extra factor $\prod_{v=1}^{p} |x-x_v|^{2\lambda_v}$ in the generalized Jacobi weight, should have an extra factor $\prod_{v=1}^{p} (z-x_v)^{2\lambda_v}$.

Similarly we have constructed in [16, Section 6] a parametrix P_{-1} in the disk $U_{\delta,-1}$ with radius $\delta>0$ and center -1. This is a matrix valued function in $U_{\delta,-1}$ that has the same jumps as S on Σ, that matches with N on $\partial U_{\delta,-1}$

$$P_{-1}(z)N^{-1}(z) = I + O(1/n), \quad \text{as } n\to\infty, \text{ uniformly for } z\in\partial U_{\delta,-1}, \tag{3.23}$$

and that has the same behavior as $S(z)$ near $z=-1$. The parametrix is given in [16, Section 6], and is constructed out of Bessel functions of order β. We note that the scalar function $\tilde{W}$ in [16, (6.52)] should have an extra factor $\prod_{v=1}^{p} (x_v-z)^{2\lambda_v}$.

We also have to construct a local parametrix P_{x_v} near the algebraic singularities x_v. Let U_{δ,x_v} be the disk, with center x_v and radius $\delta>0$ so that the closures of the disks $U_{\delta,x_0},\dots,U_{\delta,x_{p+1}}$ do not intersect and so that all the disks lie in U. The construction of the parametrix P_{x_v} will be done in Section 4. For now, let us assume that we have a 2×2 matrix valued function P_{x_v} with the same jumps as S, that matches with N on $\partial U_{\delta,x_v}$,

$$P_{x_v}(z)N^{-1}(z) = I + O(1/n), \quad \text{as } n\to\infty, \text{ uniformly for } z\in\partial U_{\delta,x_v}, \tag{3.24}$$

and that has the same behavior as $S(z)$ near $z=x_v$.

3.3. *Third transformation $S\mapsto R$*

Using the parametrix for the outside region, the parametrices near the endpoints, and the parametrices near the algebraic singularities we do the final transformation. Let us define the matrix valued function R as

$$R(z) = \begin{cases} S(z)N^{-1}(z), & \text{for } z\in\mathbb{C}\setminus\left(\Sigma\cup\left[\bigcup_{v=0}^{p+1} U_{\delta,x_v}\right]\right), \\ S(z)P_{x_v}^{-1}(z), & \text{for } z\in U_{\delta,x_v}\setminus\Sigma, \text{ and } v=0,\dots,p+1. \end{cases} \tag{3.25}$$

Remark 3.5. Note that the inverses of the parametrices exist. For N, P_{-1} and P_1 this was already known, see [16]. In the next section we will show that P_{x_v} is also invertible.

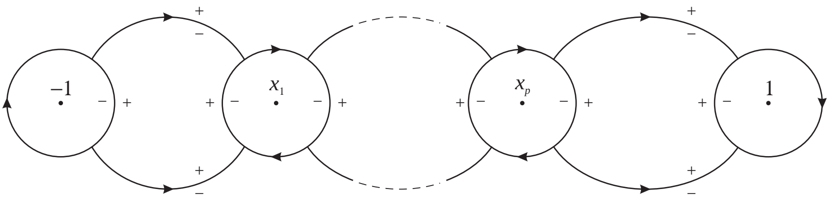

Fig. 3. The reduced system of contours Σ_R with circles U_{δ,x_ν} of radius δ and center x_ν.

If we take care of the behavior near the algebraic singularities in the same ;way as near the endpoints, it turns out that R satisfies the following RH problem, cf. [16, Section 7], with jumps on the reduced system of contours Σ_R, see Fig. 3.

RH problem for R.

(a) $R(z)$ is analytic for $z\in\mathbb{C}\setminus\Sigma_R$.
(b) $R(z)$ satisfies the following jump relations on Σ_R:

$$R_+(z) = R_-(z)P_{x_\nu}(z)N^{-1}(z), \qquad \text{for } z\in\partial U_{\delta,x_\nu}, \text{ and } \nu = 0,\dots,p+1, \tag{3.26}$$

$$R_+(z) = R_-(z)N(z)\begin{pmatrix} 1 & 0 \\ w(z)^{-1}\varphi(z)^{-2n} & 1 \end{pmatrix}N^{-1}(z), \qquad \text{for } z\in\Sigma_R\setminus\left(\bigcup_{\nu=0}^{p+1}\partial U_{\delta,x_\nu}\right). \tag{3.27}$$

(c) $R(z)$ has the following behavior at infinity:

$$R(z) = I + O(1/z), \quad \text{as } z\to\infty. \tag{3.28}$$

By (3.22)–(3.24), the jump matrices on the circles are uniformly close to the identity matrix as $n\to\infty$. On the lips of the lens we have by (3.27), as in [16, Section 7], that the jump matrix converges uniformly to the identity matrix at an exponential rate. So, all jump matrices are uniformly close to the identity matrix. This implies that, cf. [5,7],

$$R(z) = I + O(1/n), \quad \text{as } n\to\infty, \text{ uniformly for } z\in\mathbb{C}\setminus\Sigma_R. \tag{3.29}$$

Remark 3.6. We will show in Section 5 that the $O(1/n)$ term in (3.29) can be developed into a complete asymptotic expansion in powers of $1/n$. This expansion will be used to prove Theorem 1.1.

4. Parametrix near the algebraic singularity x_ν

Fix $\nu \in \{1, \dots, p\}$. In this section we construct a 2×2 matrix valued function P_{x_ν} that satisfies the following RH problem.

RH problem for P_{x_ν}.

(a) $P_{x_\nu}(z)$ is defined and analytic for $z \in U_{\delta_0, x_\nu} \setminus \Sigma$ for some $\delta_0 > \delta$.
(b) $P_{x_\nu}(z)$ satisfies the following jump relations on $\Sigma^o \cap U_{\delta, x_\nu}$:

$$P_{x_\nu,+}(z) = P_{x_\nu,-}(z) \begin{pmatrix} 1 & 0 \\ w(z)^{-1}\varphi(z)^{-2n} & 1 \end{pmatrix}, \quad \text{for } z \in (\Sigma^o \cap \mathbb{C}_\pm) \cap U_{\delta, x_\nu}, \tag{4.1}$$

$$P_{x_\nu,+}(x) = P_{x_\nu,-}(x) \begin{pmatrix} 0 & w(x) \\ -w(x)^{-1} & 0 \end{pmatrix}, \quad \text{for } x \in (\Sigma^o \cap (-1,1)) \cap U_{\delta, x_\nu}. \tag{4.2}$$

(c) On $\partial U_{\delta, x_\nu}$ we have, as $n \to \infty$

$$P_{x_\nu}(z) N^{-1}(z) = I + O(1/n), \quad \text{uniformly for } z \in \partial U_{\delta, x_\nu} \setminus \Sigma. \tag{4.3}$$

(d) For $\lambda_\nu < 0$, $P_{x_\nu}(z)$ has the following behavior as $z \to x_\nu$:

$$P_{x_\nu}(z) = O\begin{pmatrix} 1 & |z - x_\nu|^{2\lambda_\nu} \\ 1 & |z - x_\nu|^{2\lambda_\nu} \end{pmatrix}, \quad \text{as } z \to x_\nu. \tag{4.4}$$

For $\lambda_\nu > 0$, $P_{x_\nu}(z)$ has the following behavior as $z \to x_\nu$:

$$P_{x_\nu}(z) = \begin{cases} O\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, & \text{as } z \to x_\nu \text{ from outside the lens,} \\ O\begin{pmatrix} |z - x_\nu|^{-2\lambda_\nu} & 1 \\ |z - x_\nu|^{-2\lambda_\nu} & 1 \end{pmatrix}, & \text{as } z \to x_\nu \text{ from inside the lens.} \end{cases} \tag{4.5}$$

We will work as follows. First, we construct a matrix valued function that satisfies conditions (a), (b) and (d) of the RH problem for P_{x_ν}. For this purpose, we will transform (in Section 4.1) this RH problem into a RH problem for $P_{x_\nu}^{(1)}$ with constant jump matrices and construct (in Section 4.2) a solution of the RH problem for $P_{x_\nu}^{(1)}$. Afterwards, we will also consider (in Section 4.3) the matching condition (c) of the RH problem for P_{x_ν}.

4.1. Transformation to a RH problem with constant jump matrices

Since h is analytic in U with positive real part, the scalar function

$$W_{x_\nu}(z) = (1-z)^{\alpha/2}(1+z)^{\beta/2}h^{1/2}(z)\prod_{k=1}^{\nu-1}(z-x_k)^{\lambda_k}\prod_{l=\nu+1}^{p}(x_l-z)^{\lambda_l} \times \begin{cases} (z-x_\nu)^{\lambda_\nu}, & \text{for } z\in(K_{x_\nu}^{\mathrm{l}}\cap U)\backslash\mathbb{R}, \\ (x_\nu-z)^{\lambda_\nu}, & \text{for } z\in(K_{x_\nu}^{\mathrm{r}}\cap U)\backslash\mathbb{R}, \end{cases} \tag{4.6}$$

is defined and analytic for $z\in U\backslash(\mathbb{R}\cup\Gamma_{x_\nu})$. Here, we recall that $K_{x_\nu}^{\mathrm{l}}$ and $K_{x_\nu}^{\mathrm{r}}$ are the sets of all points on the left, respectively right, of Γ_{x_ν}. We seek P_{x_ν} in the form

$$P_{x_\nu}(z) = E_{n,x_\nu}(z)P_{x_\nu}^{(1)}(z)W_{x_\nu}(z)^{-\sigma_3}\varphi(z)^{-n\sigma_3}, \tag{4.7}$$

where the matrix valued function E_{n,x_ν} is analytic in a neighborhood of U_{δ,x_ν}, and E_{n,x_ν} will be determined (in Section 4.3) so that the matching condition (c) of the RH problem for P_{x_ν} is satisfied.

Since P_{x_ν} has jumps on $\Sigma\cap U_{\delta,x_\nu}$, and since W_{x_ν} has a jump on $\Gamma_{x_\nu}\cap U$, the matrix valued function $P_{x_\nu}^{(1)}$ has jumps on the contour

$$\Sigma_{x_\nu} = (\Sigma\cup\Gamma_{x_\nu})\cap U_{\delta,x_\nu},$$

see Fig. 4. The contour Σ_{x_ν} consists of eight parts, which we denote by $\Sigma_1,\dots,\Sigma_8$, as shown in Fig. 4. We write

$$\Sigma_{x_\nu}^{\mathrm{o}} = \Sigma_{x_\nu}\backslash\{x_\nu\}, \quad \text{and} \quad \Sigma_k^{\mathrm{o}} = \Sigma_k\backslash\{x_\nu\}, \quad \text{for } k=1,\dots,8.$$

In order to determine the jump matrices for $P_{x_\nu}^{(1)}$, we need some information about the scalar function W_{x_ν}. Write

$$K_{x_\nu}^{\mathrm{I}} = K_{x_\nu}^{\mathrm{r}}\cap\mathbb{C}_+, \quad K_{x_\nu}^{\mathrm{II}} = K_{x_\nu}^{\mathrm{l}}\cap\mathbb{C}_+,$$
$$K_{x_\nu}^{\mathrm{III}} = K_{x_\nu}^{\mathrm{l}}\cap\mathbb{C}_-, \quad K_{x_\nu}^{\mathrm{IV}} = K_{x_\nu}^{\mathrm{r}}\cap\mathbb{C}_-.$$

So, the sets $K_{x_\nu}^{\mathrm{I}},\dots,K_{x_\nu}^{\mathrm{IV}}$ divide the complex plane into four regions divided by the real axis and the contour Γ_{x_ν}. By (3.4) and (4.6), we have for $z\in(K_{x_{\nu-1}}^{\mathrm{r}}\cap K_{x_{\nu+1}}^{\mathrm{l}})\cap U$,

$$W_{x_\nu}^2(z) = \begin{cases} w(z)e^{-2\pi i\lambda_\nu}, & \text{if } z\in K_{x_\nu}^{\mathrm{I}}\cup K_{x_\nu}^{\mathrm{III}}, \\ w(z)e^{2\pi i\lambda_\nu}, & \text{if } z\in K_{x_\nu}^{\mathrm{II}}\cup K_{x_\nu}^{\mathrm{IV}}. \end{cases} \tag{4.8}$$

Here, we recall that $K_{x_0}^{\mathrm{r}} = K_{x_{p+1}}^{\mathrm{l}} = \mathbb{C}$. From this we see that

$$W_{x_\nu,+}(x)W_{x_\nu,-}(x) = w(x), \quad \text{for } x\in(x_{\nu-1},x_{\nu+1})\backslash\{x_\nu\}. \tag{4.9}$$

By (4.6) we have on the contour Γ_{x_ν},

$$W_{x_\nu,+}(z)W_{x_\nu,-}(z)^{-1} = e^{\lambda_\nu\pi i}, \quad \text{for } z\in\Gamma_{x_\nu}\cap U. \tag{4.10}$$

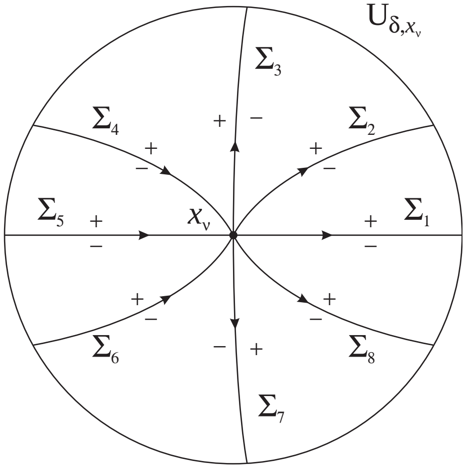

Fig. 4. The contour Σ_{x_v}. Here, $\Sigma_3 \cup \Sigma_7$ is the part of Γ_{x_v} in U_{δ,x_v}, and the remainder is the part of Σ in U_{δ,x_v}.

We now have enough information about W_{x_v} to determine the jump matrices for $P^{(1)}_{x_v}$. First, we determine the jump matrix on the lips of the lens. By (4.1), (4.7) and (4.8) the matrix valued function $P^{(1)}_{x_v}$ should satisfy on $\Sigma_2^o \cup \Sigma_4^o \cup \Sigma_6^o \cup \Sigma_8^o$ the jump relation

$$\begin{aligned} P^{(1)}_{x_v,+}(z) &= P^{(1)}_{x_v,-}(z)\varphi(z)^{-n\sigma_3} W_{x_v}(z)^{-\sigma_3} \begin{pmatrix} 1 & 0 \\ w(z)^{-1}\varphi(z)^{-2n} & 1 \end{pmatrix} W_{x_v}(z)^{\sigma_3}\varphi(z)^{n\sigma_3} \\ &= P^{(1)}_{x_v,-}(z) \begin{pmatrix} 1 & 0 \\ w(z)^{-1} W^2_{x_v}(z) & 1 \end{pmatrix} \\ &= P^{(1)}_{x_v,-}(z) \begin{pmatrix} 1 & 0 \\ e^{\pm 2\pi i \lambda_v} & 1 \end{pmatrix}, \end{aligned} \tag{4.11}$$

where in $e^{\pm 2\pi i\lambda_v}$ the $+$ sign holds for $z \in \Sigma_4^o \cup \Sigma_8^o$ and the $-$ sign for $z \in \Sigma_2^o \cup \Sigma_6^o$.

Next, we determine the jump matrix on the interval. For $x \in \Sigma_1^o \cup \Sigma_5^o$ we have by (4.2), (4.7), (4.9), and the fact that $\varphi_+(x)\varphi_-(x) = 1$,

$$\begin{aligned} P^{(1)}_{x_v,+}(x) &= P^{(1)}_{x_v,-}(x) W_{x_v,-}(x)^{-\sigma_3}\varphi_-(x)^{-n\sigma_3} \begin{pmatrix} 0 & w(x) \\ -w(x)^{-1} & 0 \end{pmatrix} \\ &\quad \times W_{x_v,+}(x)^{\sigma_3}\varphi_+(x)^{n\sigma_3} \\ &= P^{(1)}_{x_v,-}(x) \begin{pmatrix} 0 & w(x) W_{x_v,+}(x)^{-1} W_{x_v,-}(x)^{-1} \\ -w(x)^{-1} W_{x_v,+}(x) W_{x_v,-}(x) & 0 \end{pmatrix} \\ &= P^{(1)}_{x_v,-}(x) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \end{aligned} \tag{4.12}$$

And finally, we determine the jump matrix on the contour that goes vertically through x_v. For $z \in \Sigma_3^o \cup \Sigma_7^o$ we have by (4.7) and (4.10),

$$P^{(1)}_{x_v,+}(z) = P^{(1)}_{x_v,-}(z) W_{x_v,-}(z)^{-\sigma_3} W_{x_v,+}(z)^{\sigma_3} = P^{(1)}_{x_v,-}(z) e^{\lambda_v \pi i \sigma_3}. \tag{4.13}$$

We then see that we must look for a matrix valued function $P^{(1)}_{x_v}$ that satisfies the following RH problem.

RH problem for $P^{(1)}_{x_v}$.

(a) $P^{(1)}_{x_v}(z)$ is defined and analytic for $z \in U_{\delta_0,x_v} \setminus (\Sigma \cup \Gamma_{x_v})$ for some $\delta_0 > \delta$.

(b) $P^{(1)}_{x_v}(z)$ satisfies the following jump relations on $\Sigma^o_{x_v}$:

$$P^{(1)}_{x_v,+}(x) = P^{(1)}_{x_v,-}(x) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \text{for } x \in \Sigma_1^o \cup \Sigma_5^o, \tag{4.14}$$

$$P^{(1)}_{x_v,+}(z) = P^{(1)}_{x_v,-}(z) \begin{pmatrix} 1 & 0 \\ e^{-2\pi i \lambda_v} & 1 \end{pmatrix}, \quad \text{for } z \in \Sigma_2^o \cup \Sigma_6^o, \tag{4.15}$$

$$P^{(1)}_{x_v,+}(z) = P^{(1)}_{x_v,-}(z) e^{\lambda_v \pi i \sigma_3}, \quad \text{for } z \in \Sigma_3^o \cup \Sigma_7^o, \tag{4.16}$$

$$P^{(1)}_{x_v,+}(z) = P^{(1)}_{x_v,-}(z) \begin{pmatrix} 1 & 0 \\ e^{2\pi i \lambda_v} & 1 \end{pmatrix}, \quad \text{for } z \in \Sigma_4^o \cup \Sigma_8^o. \tag{4.17}$$

(c) For $\lambda_v < 0$, $P^{(1)}_{x_v}(z)$ has the following behavior as $z \to x_v$:

$$P^{(1)}_{x_v}(z) = O\begin{pmatrix} |z-x_v|^{\lambda_v} & |z-x_v|^{\lambda_v} \\ |z-x_v|^{\lambda_v} & |z-x_v|^{\lambda_v} \end{pmatrix}, \quad \text{as } z \to x_v. \tag{4.18}$$

For $\lambda_v > 0$, $P^{(1)}_{x_v}(z)$ has the following behavior as $z \to x_v$:

$$P^{(1)}_{x_v}(z) = \begin{cases} O\begin{pmatrix} |z-x_v|^{\lambda_v} & |z-x_v|^{-\lambda_v} \\ |z-x_v|^{\lambda_v} & |z-x_v|^{-\lambda_v} \end{pmatrix}, & \text{as } z \to x_v \text{ from outside the lens,} \\ O\begin{pmatrix} |z-x_v|^{-\lambda_v} & |z-x_v|^{-\lambda_v} \\ |z-x_v|^{-\lambda_v} & |z-x_v|^{-\lambda_v} \end{pmatrix}, & \text{as } z \to x_v \text{ from inside the lens.} \end{cases} \tag{4.19}$$

Remark 4.1. Condition (c) follows from condition (d) of the RH problem for P_{x_v}, since

$$P_{x_v}(z) = E_{n,x_v}(z) P^{(1)}_{x_v}(z) W_{x_v}(z)^{-\sigma_3} \varphi(z)^{-n\sigma_3},$$

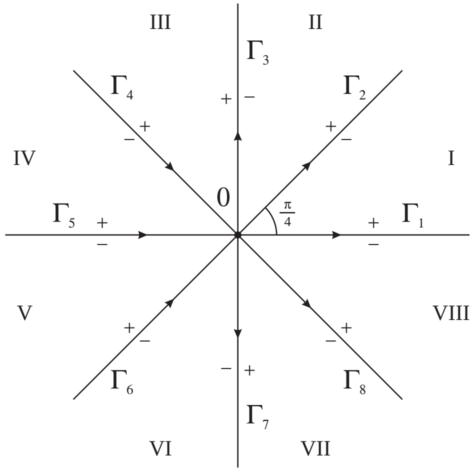

Fig. 5. The contour Σ_Ψ.

where $\varphi(z)$ is bounded and bounded away from 0 near $z = x_v$, and where $W_{x_v}(z)$ behaves like $c|z - x_v|^{\lambda_v}$ as $z \to x_v$, with a non-zero constant c.

4.2. Construction of $P^{(1)}_{x_v}$

The construction of $P^{(1)}_{x_v}$ is based upon an auxiliary RH problem for Ψ_λ in the ζ-plane with jumps on the contour $\Sigma_\Psi = \bigcup_{i=1}^{8} \Gamma_i$ consisting of eight straight rays, oriented as in Fig. 5, which divides the complex plane into eight regions I–VIII, also shown in Fig. 5. We let $\lambda > -\frac{1}{2}$.

RH problem for Ψ_λ.

(a) $\Psi_\lambda(\zeta)$ is analytic for $\zeta \in \mathbb{C} \setminus \Sigma_\Psi$.
(b) $\Psi_\lambda(\zeta)$ satisfies the following jump relations on Σ_Ψ:

$$\Psi_{\lambda,+}(\zeta) = \Psi_{\lambda,-}(\zeta) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \text{for } \zeta \in \Gamma_1 \cup \Gamma_5, \tag{4.20}$$

$$\Psi_{\lambda,+}(\zeta) = \Psi_{\lambda,-}(\zeta) \begin{pmatrix} 1 & 0 \\ e^{-2\pi i\lambda} & 1 \end{pmatrix}, \quad \text{for } \zeta \in \Gamma_2 \cup \Gamma_6, \tag{4.21}$$

$$\Psi_{\lambda,+}(\zeta) = \Psi_{\lambda,-}(\zeta) e^{\lambda\pi i\sigma_3}, \quad \text{for } \zeta \in \Gamma_3 \cup \Gamma_7, \tag{4.22}$$

$$\Psi_{\lambda,+}(\zeta) = \Psi_{\lambda,-}(\zeta) \begin{pmatrix} 1 & 0 \\ e^{2\pi i\lambda} & 1 \end{pmatrix}, \quad \text{for } \zeta \in \Gamma_4 \cup \Gamma_8. \tag{4.23}$$

(c) For $\lambda<0$, $\Psi_\lambda(\zeta)$ has the following behavior as $\zeta\to 0$:

$$\Psi_\lambda(\zeta)=O\begin{pmatrix}|\zeta|^{\lambda} & |\zeta|^{\lambda}\\ |\zeta|^{\lambda} & |\zeta|^{\lambda}\end{pmatrix},\quad \text{as } \zeta\to 0. \tag{4.24}$$

For $\lambda>0$, $\Psi_\lambda(\zeta)$ has the following behavior as $\zeta\to 0$:

$$\Psi_{\lambda_v}(\zeta)=\begin{cases} O\begin{pmatrix}|\zeta|^{\lambda} & |\zeta|^{-\lambda}\\ |\zeta|^{\lambda} & |\zeta|^{-\lambda}\end{pmatrix}, & \text{as } \zeta\to 0 \text{ for } \zeta\in \text{ II, III, VI, VII},\\ O\begin{pmatrix}|\zeta|^{-\lambda} & |\zeta|^{-\lambda}\\ |\zeta|^{-\lambda} & |\zeta|^{-\lambda}\end{pmatrix}, & \text{as } \zeta\to 0 \text{ for } \zeta\in \text{ I, IV, V, VIII}.\end{cases} \tag{4.25}$$

We construct a solution Ψ_λ of this RH problem out of the modified Bessel functions $I_{\lambda\pm 1/2}$ and $K_{\lambda\pm 1/2}$, and out of the Hankel functions $H^{(1)}_{\lambda\pm 1/2}$ and $H^{(2)}_{\lambda\pm 1/2}$. For $\zeta\in$ I, we define $\Psi_\lambda(\zeta)$ by

$$\Psi_\lambda(\zeta)=\frac{1}{2}\sqrt{\pi}\zeta^{1/2}\begin{pmatrix}H^{(2)}_{\lambda+\frac{1}{2}}(\zeta) & -iH^{(1)}_{\lambda+\frac{1}{2}}(\zeta)\\ H^{(2)}_{\lambda-\frac{1}{2}}(\zeta) & -iH^{(1)}_{\lambda-\frac{1}{2}}(\zeta)\end{pmatrix}e^{-(\lambda+\frac{1}{4})\pi i\sigma_3}. \tag{4.26}$$

For $\zeta\in$ II, by

$$\Psi_\lambda(\zeta)=\begin{pmatrix}\sqrt{\pi}\zeta^{1/2}I_{\lambda+\frac{1}{2}}(\zeta e^{-\frac{\pi i}{2}}) & -\frac{1}{\sqrt{\pi}}\zeta^{1/2}K_{\lambda+\frac{1}{2}}(\zeta e^{-\frac{\pi i}{2}})\\ -i\sqrt{\pi}\zeta^{1/2}I_{\lambda-\frac{1}{2}}(\zeta e^{-\frac{\pi i}{2}}) & -\frac{i}{\sqrt{\pi}}\zeta^{1/2}K_{\lambda-\frac{1}{2}}(\zeta e^{-\frac{\pi i}{2}})\end{pmatrix}e^{-\frac{1}{2}\lambda\pi i\sigma_3}. \tag{4.27}$$

For $\zeta\in$ III, by

$$\Psi_\lambda(\zeta)=\begin{pmatrix}\sqrt{\pi}\zeta^{1/2}I_{\lambda+\frac{1}{2}}(\zeta e^{-\frac{\pi i}{2}}) & -\frac{1}{\sqrt{\pi}}\zeta^{1/2}K_{\lambda+\frac{1}{2}}(\zeta e^{-\frac{\pi i}{2}})\\ -i\sqrt{\pi}\zeta^{1/2}I_{\lambda-\frac{1}{2}}(\zeta e^{-\frac{\pi i}{2}}) & -\frac{i}{\sqrt{\pi}}\zeta^{1/2}K_{\lambda-\frac{1}{2}}(\zeta e^{-\frac{\pi i}{2}})\end{pmatrix}e^{\frac{1}{2}\lambda\pi i\sigma_3}. \tag{4.28}$$

For $\zeta\in$ IV, by

$$\Psi_\lambda(\zeta)=\frac{1}{2}\sqrt{\pi}(-\zeta)^{1/2}\begin{pmatrix}iH^{(1)}_{\lambda+\frac{1}{2}}(-\zeta) & -H^{(2)}_{\lambda+\frac{1}{2}}(-\zeta)\\ -iH^{(1)}_{\lambda-\frac{1}{2}}(-\zeta) & H^{(2)}_{\lambda-\frac{1}{2}}(-\zeta)\end{pmatrix}e^{(\lambda+\frac{1}{4})\pi i\sigma_3}. \tag{4.29}$$

For $\zeta\in$ V, by

$$\Psi_\lambda(\zeta)=\frac{1}{2}\sqrt{\pi}(-\zeta)^{1/2}\begin{pmatrix}-H^{(2)}_{\lambda+\frac{1}{2}}(-\zeta) & -iH^{(1)}_{\lambda+\frac{1}{2}}(-\zeta)\\ H^{(2)}_{\lambda-\frac{1}{2}}(-\zeta) & iH^{(1)}_{\lambda-\frac{1}{2}}(-\zeta)\end{pmatrix}e^{-(\lambda+\frac{1}{4})\pi i\sigma_3}. \tag{4.30}$$

For $\zeta \in$ VI, by

$$\Psi_\lambda(\zeta) = \begin{pmatrix} -i\sqrt{\pi}\zeta^{1/2} I_{\lambda+\frac{1}{2}}(\zeta e^{\frac{\pi i}{2}}) & -\frac{i}{\sqrt{\pi}}\zeta^{1/2} K_{\lambda+\frac{1}{2}}(\zeta e^{\frac{\pi i}{2}}) \\ \sqrt{\pi}\zeta^{1/2} I_{\lambda-\frac{1}{2}}(\zeta e^{\frac{\pi i}{2}}) & -\frac{1}{\sqrt{\pi}}\zeta^{1/2} K_{\lambda-\frac{1}{2}}(\zeta e^{\frac{\pi i}{2}}) \end{pmatrix} e^{-\frac{1}{2}\lambda\pi i\sigma_3}. \tag{4.31}$$

For $\zeta \in$ VII, by

$$\Psi_\lambda(\zeta) = \begin{pmatrix} -i\sqrt{\pi}\zeta^{1/2} I_{\lambda+\frac{1}{2}}(\zeta e^{\frac{\pi i}{2}}) & -\frac{i}{\sqrt{\pi}}\zeta^{1/2} K_{\lambda+\frac{1}{2}}(\zeta e^{\frac{\pi i}{2}}) \\ \sqrt{\pi}\zeta^{1/2} I_{\lambda-\frac{1}{2}}(\zeta e^{\frac{\pi i}{2}}) & -\frac{1}{\sqrt{\pi}}\zeta^{1/2} K_{\lambda-\frac{1}{2}}(\zeta e^{\frac{\pi i}{2}}) \end{pmatrix} e^{\frac{1}{2}\lambda\pi i\sigma_3}. \tag{4.32}$$

And finally, for $\zeta \in$ VIII, we define it by

$$\Psi_\lambda(\zeta) = \frac{1}{2}\sqrt{\pi}\zeta^{1/2} \begin{pmatrix} -iH^{(1)}_{\lambda+\frac{1}{2}}(\zeta) & -H^{(2)}_{\lambda+\frac{1}{2}}(\zeta) \\ -iH^{(1)}_{\lambda-\frac{1}{2}}(\zeta) & -H^{(2)}_{\lambda-\frac{1}{2}}(\zeta) \end{pmatrix} e^{(\lambda+\frac{1}{4})\pi i\sigma_3}. \tag{4.33}$$

Theorem 4.2. *The matrix valued function* Ψ_λ, *defined by* (4.26)–(4.33), *is a solution of the RH problem for* Ψ_λ.

Proof. The functions $I_{\lambda\pm1/2}, K_{\lambda\pm1/2}, H^{(1)}_{\lambda\pm1/2}$ and $H^{(2)}_{\lambda\pm1/2}$ are defined and analytic in the complex plane with a branch cut along the negative real axis. So, the matrix valued function Ψ_λ defined by (4.26)–(4.33) is analytic in the respective regions, and condition (a) of the RH problem is therefore satisfied. Condition (c) follows easily from [1, Formulas 9.1.9, 9.6.7 and 9.6.9]. So, it remains to prove that jump conditions (4.20)–(4.23) are satisfied.

Jump conditions (4.20) *and* (4.22): By inspection, it is easy to see that these jump conditions are satisfied.

Jump condition (4.21) *for* $\zeta \in \Gamma_2$: We use (4.27) to evaluate $\Psi_{\lambda,+}(\zeta)$ and (4.26) to evaluate $\Psi_{\lambda,-}(\zeta)$. From (4.26) and [1, Formulas 9.1.3, 9.1.4 and 9.6.3], the 1,1-entry and the 2,1-entry on the right of (4.21) are equal to

$$\begin{aligned} &\Psi_{\lambda,11,-}(\zeta) + e^{-2\pi i\lambda}\Psi_{\lambda,12,-}(\zeta) \\ &\quad = \frac{1}{2}\sqrt{\pi}\zeta^{1/2}e^{-(\lambda+\frac{1}{4})\pi i}H^{(2)}_{\lambda+\frac{1}{2}}(\zeta) + \frac{1}{2}\sqrt{\pi}\zeta^{1/2}e^{-(\lambda+\frac{1}{4})\pi i}H^{(1)}_{\lambda+\frac{1}{2}}(\zeta) \\ &\quad = \sqrt{\pi}\zeta^{1/2}e^{-(\lambda+\frac{1}{4})\pi i}J_{\lambda+\frac{1}{2}}(\zeta) \\ &\quad = \sqrt{\pi}\zeta^{1/2}I_{\lambda+\frac{1}{2}}(\zeta e^{-\frac{\pi i}{2}})e^{-\frac{1}{2}\lambda\pi i}, \end{aligned} \tag{4.34}$$

and

$$\begin{aligned}
&\Psi_{\lambda,21,-}(\zeta)+e^{-2\pi i\lambda}\Psi_{\lambda,22,-}(\zeta)\\
&\quad=\frac{1}{2}\sqrt{\pi}\zeta^{1/2}e^{-(\lambda+\frac{1}{4})\pi i}H^{(2)}_{\lambda-\frac{1}{2}}(\zeta)+\frac{1}{2}\sqrt{\pi}\zeta^{1/2}e^{-(\lambda+\frac{1}{4})\pi i}H^{(1)}_{\lambda-\frac{1}{2}}(\zeta)\\
&\quad=\sqrt{\pi}\zeta^{1/2}e^{-(\lambda+\frac{1}{4})\pi i}J_{\lambda-\frac{1}{2}}(\zeta)\\
&\quad=-i\sqrt{\pi}\zeta^{1/2}I_{\lambda-\frac{1}{2}}(\zeta e^{-\frac{\pi i}{2}})e^{-\frac{1}{2}\lambda\pi i},
\end{aligned}\tag{4.35}$$

respectively. By (4.27) we then see that the first columns of both sides of (4.21) agree. From (4.26), (4.27) and [1, Formula 9.6.4], the second columns of both sides of (4.21) agree as well.

Jump condition (4.21) *for* $\zeta\in\Gamma_6$: We use (4.30) to evaluate $\Psi_{\lambda,+}(\zeta)$ and (4.31) to evaluate $\Psi_{\lambda,-}(\zeta)$. Since $-\zeta=\zeta e^{\pi i}$, we have, from (4.31) and [1, Formula 9.6.4], that the 1,2-entry and the 2,2-entry on the right of (4.21) are equal to

$$\Psi_{\lambda,12,-}(\zeta)=-\frac{i}{\sqrt{\pi}}\zeta^{1/2}e^{\frac{1}{2}\lambda\pi i}K_{\lambda+\frac{1}{2}}(\zeta e^{\frac{\pi i}{2}})=-\frac{i}{2}\sqrt{\pi}(-\zeta)^{1/2}H^{(1)}_{\lambda+\frac{1}{2}}(-\zeta)e^{(\lambda+\frac{1}{4})\pi i},\tag{4.36}$$

and

$$\Psi_{\lambda,22,-}(\zeta)=-\frac{1}{\sqrt{\pi}}\zeta^{1/2}e^{\frac{1}{2}\lambda\pi i}K_{\lambda-\frac{1}{2}}(\zeta e^{\frac{\pi i}{2}})=\frac{i}{2}\sqrt{\pi}(-\zeta)^{1/2}H^{(1)}_{\lambda-\frac{1}{2}}(-\zeta)e^{(\lambda+\frac{1}{4})\pi i},\tag{4.37}$$

respectively. So, by (4.30) we see that the second columns of both sides of (4.21) agree. Since $-\zeta=\zeta e^{\pi i}$ we have, from (4.31), (4.36), (4.37) and [1, Formulas 9.1.3, 9.1.4 and 9.6.3], that the 1,1-entry and the 2,1-entry on the right of (4.21) are equal to

$$\begin{aligned}
&\Psi_{\lambda,11,-}(\zeta)+e^{-2\pi i\lambda}\Psi_{\lambda,12,-}(\zeta)\\
&\quad=-\sqrt{\pi}(-\zeta)^{1/2}e^{-\frac{1}{2}\lambda\pi i}I_{\lambda+\frac{1}{2}}(\zeta e^{\frac{\pi i}{2}})+\frac{1}{2}\sqrt{\pi}(-\zeta)^{1/2}e^{-(\lambda+\frac{1}{4})\pi i}H^{(1)}_{\lambda+\frac{1}{2}}(-\zeta)\\
&\quad=-\sqrt{\pi}(-\zeta)^{1/2}e^{-(\lambda+\frac{1}{4})\pi i}J_{\lambda+\frac{1}{2}}(-\zeta)+\frac{1}{2}\sqrt{\pi}(-\zeta)^{1/2}e^{-(\lambda+\frac{1}{4})\pi i}H^{(1)}_{\lambda+\frac{1}{2}}(-\zeta)\\
&\quad=-\frac{1}{2}\sqrt{\pi}(-\zeta)^{1/2}H^{(2)}_{\lambda+\frac{1}{2}}(-\zeta)e^{-(\lambda+\frac{1}{4})\pi i},
\end{aligned}\tag{4.38}$$

and

$$\begin{aligned}
&\Psi_{\lambda,21,-}(\zeta)+e^{-2\pi i\lambda}\Psi_{\lambda,22,-}(\zeta)\\
&\quad=-i\sqrt{\pi}(-\zeta)^{1/2}e^{-\frac{1}{2}\lambda\pi i}I_{\lambda-\frac{1}{2}}(\zeta e^{\frac{\pi i}{2}})-\frac{1}{2}\sqrt{\pi}(-\zeta)^{1/2}e^{-(\lambda+\frac{1}{4})\pi i}H^{(1)}_{\lambda-\frac{1}{2}}(-\zeta)\\
&\quad=\sqrt{\pi}(-\zeta)^{1/2}e^{-(\lambda+\frac{1}{4})\pi i}J_{\lambda-\frac{1}{2}}(-\zeta)-\frac{1}{2}\sqrt{\pi}(-\zeta)^{1/2}e^{-(\lambda+\frac{1}{4})\pi i}H^{(1)}_{\lambda-\frac{1}{2}}(-\zeta)\\
&\quad=\frac{1}{2}\sqrt{\pi}(-\zeta)^{1/2}H^{(2)}_{\lambda-\frac{1}{2}}(-\zeta)e^{-(\lambda+\frac{1}{4})\pi i},
\end{aligned}\tag{4.39}$$

respectively. By (4.30) we then see that the first columns of both sides of (4.21) agree as well. We now have proven that jump condition (4.21) is satisfied.

Jump condition (4.23): Similarly, we can prove that this jump condition is also satisfied. Here, we also use [1, Formula 9.1.35], and the details are left to the reader. This implies that the theorem is proved. □

Now, we explain how we get $P^{(1)}_{x_v}$ out of the solution Ψ_{λ_v} (depending on the parameter λ_v) of the RH problem for Ψ_{λ_v}. We make use of the following scalar function,

$$f_{x_v}(z) = \begin{cases} i\log\varphi(z) - i\log\varphi_+(x_v), & \text{for } \operatorname{Im} z > 0, \\ -i\log\varphi(z) - i\log\varphi_+(x_v), & \text{for } \operatorname{Im} z < 0, \end{cases} \tag{4.40}$$

which is defined and analytic for $z \in \mathbb{C}\backslash\mathbb{R}$. For $x \in (-1,1)$ we have, since $\varphi_+(x)\varphi_-(x) = 1$, that $f_{x_v,+}(x) = f_{x_v,-}(x)$, so that f_{x_v} is also analytic across the interval $(-1,1)$. The behavior of f_{x_v} near x_v is

$$f_{x_v}(z) = \frac{1}{\sqrt{1-x_v^2}}(z-x_v) + O((z-x_v)^2), \quad \text{as } z \to x_v.$$

Since f_{x_v} is analytic near x_v, and since $f'(x_v) \neq 0$, the scalar function f_{x_v} is a one-to-one conformal mapping on a neighborhood of x_v. So, if we choose $\delta > 0$ sufficiently small, f_{x_v} is a one-to-one conformal mapping on U_{δ,x_v} and the image of U_{δ,x_v} under the mapping $\zeta = f_{x_v}$ is convex.

For $x \in (-1,1)$ we have by (4.40) that $f_{x_v}(x) = \arccos x_v - \arccos x$. So, $f_{x_v}(x)$ is real for $x \in (-1,1)$. If $x > x_v$ we have $f_{x_v}(x) > 0$, and if $x < x_v$ we have $f_{x_v}(x) < 0$. Since f_{x_v} is a conformal mapping, this implies that f_{x_v} maps $U_{\delta,x_v} \cap \mathbb{C}_+$ one-to-one onto $f_{x_v}(U_{\delta,x_v}) \cap \mathbb{C}_+$, and $U_{\delta,x_v} \cap \mathbb{C}_-$ one-to-one onto $f_{x_v}(U_{\delta,x_v}) \cap \mathbb{C}_-$.

We now come back to the special choice of the contour Γ_{x_v}, which we used to continuate our weight analytically, see Section 3.2. For $z \in \Gamma_{x_v} \cap \mathbb{C}_+$ we have $\arg\varphi(z) = \arccos x_v$, by construction of Γ_{x_v}, and for $z \in \Gamma_{x_v} \cap \mathbb{C}_-$ we have $\arg\varphi(z) = -\arccos x_v$. By (4.40) we then have $\operatorname{Re} f_{x_v}(z) = 0$, for $z \in \Gamma_{x_v}$. This implies that the image of the contour Γ_{x_v} under the mapping $\zeta = f_{x_v}$ is the imaginary axis, which explains our choice of Γ_{x_v}.

We remember that the contour Σ_{x_v} was not yet completely defined. Now, we define the contours $\Sigma_2 \cup \Sigma_4 \cup \Sigma_6 \cup \Sigma_8$ as the preimages of the parts of the corresponding rays $\Gamma_2 \cup \Gamma_4 \cup \Gamma_6 \cup \Gamma_8$ in $f_{x_v}(U_{\delta,x_v})$ under the mapping $\zeta = f_{x_v}(z)$, see Fig. 6. We then have immediately that we can define

$$P^{(1)}_{x_v}(z) = \Psi_{\lambda_v}(nf_{x_v}(z)), \tag{4.41}$$

and $P^{(1)}_{x_v}$ will solve the RH problem for $P^{(1)}_{x_v}$.

Remark 4.3. We can use any one-to-one conformal mapping on U_{δ,x_v} to construct $P^{(1)}_{x_v}$ out of Ψ_{λ_v}. However, we have to choose it so as to compensate for the factor $\varphi(z)^{-n\sigma_3}$ in (4.7). In the next section we will see that our choice of conformal mapping will do the job.

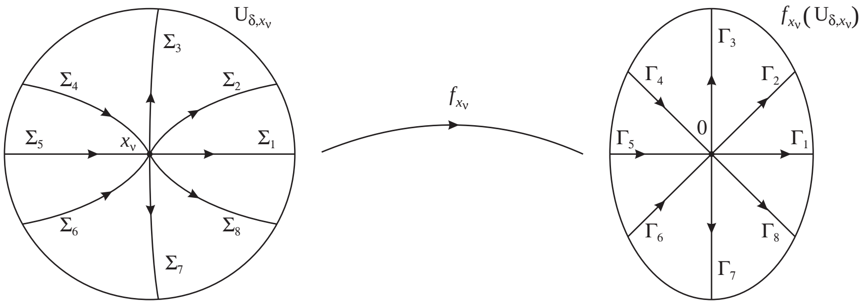

Fig. 6. The conformal mapping f_{x_v}. For every $k = 1, \dots, 8$, the contour Σ_k is mapped onto the part of the corresponding ray Γ_k in $f_{x_v}(U_{\delta,x_v})$.

4.3. Construction of E_{n,x_v}

We recall that for every matrix valued function E_{n,x_v} analytic in a neighborhood of U_{δ,x_v}, the matrix valued function P_{x_v} given by

$$P_{x_v}(z) = E_{n,x_v}(z)\Psi_{\lambda_v}(nf_{x_v}(z))W_{x_v}(z)^{-\sigma_3}\varphi(z)^{-n\sigma_3} \tag{4.42}$$

satisfies conditions (a), (b) and (d) of the RH problem for P_{x_v}. In this section we want to determine E_{n,x_v} so that the matching condition (c) is satisfied as well. To this end we need to know the asymptotic behavior of Ψ_{λ_v} at infinity, and use this to determine E_{n,x_v}. At the end of this section we also show that E_{n,x_v} is analytic in a neighborhood of U_{δ,x_v}, so that the parametrix P_{x_v} is completely defined.

In order to determine the asymptotic behavior of Ψ_{λ_v} at infinity, we insert the behavior of the Bessel functions at infinity into the matrix valued function Ψ_{λ_v}, given by (4.26)–(4.33). See [1, Formulas 9.7.1–9.7.4] for the behavior of the modified Bessel functions at infinity, and [1, Formulas 9.2.7–9.2.10] for the behavior of the Hankel functions at infinity. Then, a straightforward calculation gives us the asymptotic behavior of Ψ_{λ_v} at infinity. The behavior is different in each quadrant. For the upper half-plane we find as $\zeta \to \infty$,

$$\Psi_{\lambda_v}(\zeta) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}\left[I + O\left(\frac{1}{\zeta}\right)\right] e^{\frac{\pi i}{4}\sigma_3} e^{-i\zeta\sigma_3} e^{-\frac{1}{2}\lambda_v \pi i \sigma_3}, \tag{4.43}$$

uniformly for ζ in the first quadrant, and

$$\Psi_{\lambda_v}(\zeta) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}\left[I + O\left(\frac{1}{\zeta}\right)\right] e^{\frac{\pi i}{4}\sigma_3} e^{-i\zeta\sigma_3} e^{\frac{1}{2}\lambda_v \pi i \sigma_3}, \tag{4.44}$$

uniformly for ζ in the second quadrant. For the lower half-plane we find as $\zeta \to \infty$,

$$\Psi_{\lambda_v}(\zeta) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}\left[I + O\left(\frac{1}{\zeta}\right)\right] e^{\frac{\pi i}{4}\sigma_3} e^{-i\zeta\sigma_3} e^{\frac{1}{2}\lambda_v \pi i \sigma_3}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{4.45}$$

uniformly for ζ in the third quadrant, and

$$\Psi_{\lambda_v}(\zeta) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}\left[I + O\left(\frac{1}{\zeta}\right)\right] e^{\frac{\pi i}{4}\sigma_3} e^{-i\zeta\sigma_3} e^{-\frac{1}{2}\lambda_v \pi i \sigma_3}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{4.46}$$

uniformly for ζ in the fourth quadrant.

Now, we use the asymptotic behavior (4.43)–(4.46) of Ψ_{λ_v} at infinity to determine E_{n,x_v}. We explain this only for the region $\partial U_{\delta,x_v} \cap K^{\mathrm{I}}_{x_v}$. The other cases are similar and the details are left to the reader. For $z \in \partial U_{\delta,x_v} \cap K^{\mathrm{I}}_{x_v}$ we have, since f_{x_v} is a one-to-one conformal mapping on U_{δ,x_v}, that $nf_{x_v}(z)$ lies in the first quadrant, cf. Fig. 6. So, we may use (4.43) to evaluate the asymptotic behavior of $\Psi_{\lambda_v}(nf_{x_v}(z))$ as $n \to \infty$. Since $\operatorname{Im} z > 0$, we have by (4.40),

$$e^{-inf_{x_v}(z)} = \varphi_+(x_v)^{-n}\varphi(z)^n.$$

Using (4.42) and (4.43) we then find

$$\begin{aligned} P_{x_v}(z)N^{-1}(z) = E_{n,x_v}(z)\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}\left[I + O\left(\frac{1}{n}\right)\right] \\ \times e^{\frac{\pi i}{4}\sigma_3}\varphi_+(x_v)^{-n\sigma_3} e^{-\frac{1}{2}\lambda_v\pi i\sigma_3} W_{x_v}(z)^{-\sigma_3} N^{-1}(z), \end{aligned}$$

as $n \to \infty$, uniformly for $z \in \partial U_{\delta,x_v} \cap K^{\mathrm{I}}_{x_v}$. So, in order that the matching condition is satisfied we define for $z \in U \cap K^{\mathrm{I}}_{x_v}$,

$$E_{n,x_v}(z) = N(z)W_{x_v}(z)^{\sigma_3} e^{\frac{1}{2}\lambda_v\pi i\sigma_3}\varphi_+(x_v)^{n\sigma_3} e^{-\frac{\pi i}{4}\sigma_3}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}. \tag{4.47}$$

Remark 4.4. With this E_{n,x_v} we see that

$$\begin{aligned} P_{x_v}(z)N^{-1}(z) = N(z)W_{x_v}(z)^{\sigma_3}\varphi_+(x_v)^{n\sigma_3}\left[I + O\left(\frac{1}{n}\right)\right] \\ \times \varphi_+(x_v)^{-n\sigma_3} W_{x_v}(z)^{-\sigma_3} N^{-1}(z), \end{aligned}$$

as $n \to \infty$, uniformly for $z \in \partial U_{\delta,x_v} \cap K^{\mathrm{I}}_{x_v}$. Since $|\varphi_+(x_v)| = 1$, and since W_{x_v} as well as all entries of N are bounded and bounded away from 0 on $\partial U_{\delta,x_v}$, the matching condition is satisfied on $\partial U_{\delta,x_v} \cap K^{\mathrm{I}}_{x_v}$.

Similarly, we use (4.40) and (4.42) together with the asymptotic behavior (4.44)–(4.46) of Ψ_{λ_v} at infinity to determine E_{n,x_v} in the other regions. Straightforward calculations then show that we have to define $E_{n,x_v}(z)$ for $z \in U \setminus (\mathbb{R} \cup \Gamma_{x_v})$ as

$$E_{n,x_v}(z) = E_v(z)\varphi_+(x_v)^{n\sigma_3} e^{-\frac{\pi i}{4}\sigma_3}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}, \tag{4.48}$$

where the matrix valued function $E_v(z)$ does not depend on n, is analytic for $z \in U \setminus (\mathbb{R} \cup \Gamma_{x_v})$ and given by

$$E_v(z) = N(z)W_{x_v}(z)^{\sigma_3} e^{\frac{1}{2}\lambda_v\pi i\sigma_3}, \quad \text{for } z \in U \cap K^{\mathrm{I}}_{x_v}, \tag{4.49}$$

$$E_v(z) = N(z) W_{x_v}(z)^{\sigma_3} e^{-\frac{1}{2}\lambda_v \pi i \sigma_3}, \quad \text{for } z \in U \cap K_{x_v}^{\mathrm{II}}, \tag{4.50}$$

$$E_v(z) = N(z) W_{x_v}(z)^{\sigma_3} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} e^{-\frac{1}{2}\lambda_v \pi i \sigma_3}, \quad \text{for } z \in U \cap K_{x_v}^{\mathrm{III}}, \tag{4.51}$$

$$E_v(z) = N(z) W_{x_v}(z)^{\sigma_3} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} e^{\frac{1}{2}\lambda_v \pi i \sigma_3}, \quad \text{for } z \in U \cap K_{x_v}^{\mathrm{IV}}. \tag{4.52}$$

Now, everything is fine, except for the fact that E_{n,x_v} is analytic in $U \backslash (\mathbb{R} \cup \Gamma_{x_v})$, but we want it to be analytic in a full neighborhood of x_v. This will be proven in the next proposition.

Proposition 4.5. *The matrix valued function E_{n,x_v} defined by* (4.48)–(4.52) *is analytic in* $U \backslash ((-\infty, x_{v-1}] \cup [x_{v+1}, \infty))$.

Proof. By (4.48) it suffices to prove that E_v is analytic in $U \backslash ((-\infty, x_{v-1}] \cup [x_{v+1}, \infty))$. We will check that E_v has no jumps on $(x_{v-1}, x_{v+1}) \backslash \{x_v\}$ and $(\Gamma_{x_v} \cap U) \backslash \{x_v\}$, and in addition that the isolated singularity of E_v at x_v is removable. Let (x_{v-1}, x_{v+1}) be oriented from the left to the right, and let $\Gamma_{x_v} \cap U$ be oriented so that it points away from x_v, cf. Fig. 4.

For $x \in (x_{v-1}, x_v)$ we use (4.50) to evaluate $E_{v,+}(x)$ and (4.51) to evaluate $E_{v,-}(x)$. From (4.8) we have $W_{x_v,+}(x) = w(x)^{1/2} e^{\lambda_v \pi i}$ and $W_{x_v,-}(x) = w(x)^{1/2} e^{-\lambda_v \pi i}$. Therefore, by (3.15), (4.50) and (4.51),

$$\begin{aligned} E_{v,+}(x) &= N_-(x) \begin{pmatrix} 0 & w(x) \\ -w(x)^{-1} & 0 \end{pmatrix} w(x)^{\sigma_3/2} e^{\lambda_v \pi i \sigma_3} e^{-\frac{1}{2}\lambda_v \pi i \sigma_3} \\ &= N_-(x) w(x)^{\sigma_3/2} e^{-\lambda_v \pi i \sigma_3} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} e^{-\frac{1}{2}\lambda_v \pi i \sigma_3} \\ &= E_{v,-}(x). \end{aligned}$$

Hence, E_v is analytic across (x_{v-1}, x_v). Similarly, we have by (3.15), (4.8), (4.49) and (4.52) that E_v is analytic across (x_v, x_{v+1}) as well.

For $z \in (\Gamma_{x_v} \cap U) \cap \mathbb{C}_+$ we use (4.50) to evaluate $E_{v,+}(z)$ and (4.49) to evaluate $E_{v,-}(z)$. From (4.10) we have $W_{x_v,+}(z) W_{x_v,-}(z)^{-1} = e^{\lambda_v \pi i}$. Therefore, by (4.49) and (4.50),

$$\begin{aligned} E_{v,+}(z) &= N(z) W_{x_v,+}(z)^{\sigma_3} e^{-\frac{1}{2}\lambda_v \pi i \sigma_3} \\ &= E_{v,-}(z) e^{-\frac{1}{2}\lambda_v \pi i \sigma_3} W_{x_v,-}(z)^{-\sigma_3} W_{x_v,+}(z)^{\sigma_3} e^{-\frac{1}{2}\lambda_v \pi i \sigma_3} \\ &= E_{v,-}(z), \end{aligned}$$

so that E_v is analytic across $(\Gamma_{x_v} \cap U) \cap \mathbb{C}_+$. Similarly, we have by (4.10), (4.51) and (4.52) that E_v is analytic across $(\Gamma_{x_v} \cap U) \cap \mathbb{C}_-$ as well. We thus have proven that E_v is analytic in $U \backslash ((-\infty, x_{v-1}] \cup [x_{v+1}, \infty) \cup \{x_v\})$.

It remains to prove that the isolated singularity of E_v at x_v is removable. We have by (3.17) and (4.6) that $D(z)$ behaves like $c_1|z-x_v|^{\lambda_v}$ and $W_{x_v}(z)$ like $c_2|z-x_v|^{\lambda_v}$ as $z \to x_v$, where c_1 and c_2 are non-zero constants. Therefore

$$\frac{W_{x_v}(z)}{D(z)} = O(1), \quad \text{and} \quad \frac{D(z)}{W_{x_v}(z)} = O(1), \quad \text{as } z \to x_v.$$

So, by (3.18), each entry of $N(z)W_{x_v}(z)^{\sigma_3}$ remains bounded as $z \to x_v$. This implies by (4.49)–(4.52) that each entry of E_v remains bounded as $z \to x_v$, so that the isolated singularity of E_v at x_v is removable. Therefore, the proposition is proved. □

This ends the construction of the local parametrix near x_v.

We recall that we also wanted the local parametrix P_{x_v} to be invertible, see Remark 3.5. We will show that

$$\det P_{x_v} \equiv 1. \tag{4.53}$$

This is analogous as in [16, Section 7] and we will just give a sketch of the proof. Since E_{n,x_v} is a product of four matrices all with determinant 1, see (4.48)–(4.52), it suffices to prove from (4.42) that $\det \Psi_{\lambda_v} \equiv 1$. Using part (b) of the RH problem for Ψ_{λ_v} we find that $\det \Psi_{\lambda_v}$ is analytic in $\mathbb{C} \setminus \{0\}$. If we then use the behavior of Ψ_{λ_v} near 0 stated in part (c) of the RH problem the isolated singularity of $\det \Psi_{\lambda_v}$ at 0 has to be removable, so that $\det \Psi_{\lambda_v}$ is an entire function. Using the asymptotics of Ψ_{λ_v} at infinity given by (4.43)–(4.46) we have that $\det \Psi_{\lambda_v}(\zeta) \to 1$ as $\zeta \to \infty$. By Liouville's theorem we then have that $\det \Psi_{\lambda_v} \equiv 1$, so that also $\det P_{x_v} \equiv 1$.

5. Asymptotics of the recurrence coefficients

In this section we will determine a complete asymptotic expansion of the recurrence coefficients a_n and b_n as $n \to \infty$. Recall that a_n and b_n have been formulated in terms of the solution of the RH problem for Y, see (2.7) and (2.8). The asymptotic analysis of the RH problem for Y has been done in Section 3, and unfolding the series of transformations $Y \mapsto T \mapsto S \mapsto R$, see [16, Section 9] for details, we find

$$a_n^2 = \lim_{z \to \infty} \left(-\frac{D_\infty^2}{2i} + zR_{12}(z;n,w) \right) \left(zR_{21}(z;n,w) + \frac{1}{2iD_\infty^2} \right), \tag{5.1}$$

and

$$b_n = \lim_{z \to \infty} \, (z - zR_{11}(z;n+1,w)R_{22}(z;n,w)). \tag{5.2}$$

Remark 5.1. We note that, see [16, Lemma 8.3],

$$||R(z) - I|| = O\left(\frac{1}{n|z|}\right), \quad \text{as } n \to \infty, \tag{5.3}$$

uniformly for $|z| \geq 2$, where $\|\cdot\|$ is any matrix norm. Inserting this into (5.1) and (5.2) we find the known asymptotic behavior of the recurrence coefficients, cf. [13],

$$a_n = \frac{1}{2} + O(1/n), \quad b_n = O(1/n), \quad \text{as } n \to \infty.$$

In the rest of the paper we will develop the $O(1/n)$ terms into complete asymptotic expansions in powers of $1/n$.

In order to determine a complete asymptotic expansion of a_n and b_n we will work as follows. In Section 5.1, we will determine a complete asymptotic expansion of the jump matrix for R in powers of $1/n$ as $n \to \infty$. As a result, we obtain in Section 5.2 a complete asymptotic expansion of R. The coefficients in this expansion can be calculated explicitly via residue calculus, and we will determine the order $1/n$ term. Finally, in Section 5.3 we will use this to prove Theorem 1.1.

5.1. Asymptotic expansion of Δ

Denote the jump matrix for R as $I + \Delta$. Then, from condition (c) of the RH problem for R,

$$\Delta(z) = P_{x_v}(z)N^{-1}(z) - I, \quad \text{for } z \in \partial U_{\delta,x_v} \text{ and } v = 0, \ldots, p+1, \tag{5.4}$$

$$\Delta(z) = N(z)\begin{pmatrix} 1 & 0 \\ w(z)^{-1}\varphi(z)^{-2n} & 1 \end{pmatrix} N^{-1}(z) - I, \text{ for } z \in \Sigma_R \setminus \left(\bigcup_{v=0}^{p+1} \partial U_{\delta,x_v} \right). \tag{5.5}$$

In this section we will show that Δ has an asymptotic expansion in powers of $1/n$ of the form

$$\Delta(z) \sim \sum_{k=1}^{\infty} \frac{\Delta_k(z,n)}{n^k}, \quad \text{as } n \to \infty, \tag{5.6}$$

uniformly for $z \in \Sigma_R$, and we will also determine the coefficients $\Delta_k(z,n)$ explicitly.

Remark 5.2. The n-dependance of the coefficients in the expansion will come from the factor $\varphi_+(x_v)^{n\sigma_3}$ in the parametrices near the algebraic singularities x_v.

On the lips of the lens, Δ vanishes at an exponential rate, cf. [16, Section 7]. This implies for every k,

$$\Delta_k(z,n) = 0, \quad \text{for } z \in \Sigma_R \setminus \left(\bigcup_{v=0}^{p+1} \partial U_{\delta,x_v} \right). \tag{5.7}$$

On the circles near ± 1, the asymptotic expansion (5.6) of Δ is known, see [16, Section 8]. The restriction of $\Delta_k(\cdot,n)$ to $\partial U_{\delta,-1} \cup \partial U_{\delta,1}$ is given by [16, (8.5) and (8.6)] and does not depend on n. It has a meromorphic continuation to $U_{\delta_0,-1} \cup U_{\delta_0,1}$ for some $\delta_0 > \delta$, with poles of order at most $[(k+1)/2]$ at ± 1. For details we refer to [16, Section 8].

So, it remains to determine the asymptotic expansion of Δ on the circles near the algebraic singularities. Fix $v \in \{1, \dots, p\}$. By (4.42), (4.48) and (5.4), we have for $z \in \partial U_{\delta, x_v}$,

$$\Delta(z) = E_v(z)\varphi_+(x_v)^{n\sigma_3} e^{-\frac{\pi i}{4}\sigma_3} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \\ \times \Psi_{\lambda_v}(nf_{x_v}(z))\varphi(z)^{-n\sigma_3} W_{x_v}(z)^{-\sigma_3} N^{-1}(z) - I. \tag{5.8}$$

Here, the matrix valued function Ψ_{λ_v} is constructed out of Bessel functions, which have a complete asymptotic expansion at infinity. This implies that $\Psi_{\lambda_v}(nf_{x_v}(z))$ also has a complete asymptotic expansion as $n \to \infty$. Inserting the asymptotic expansions of the modified Bessel functions at infinity [1, Formulas 9.7.1–9.7.4] into (4.26), and the asymptotic expansions of the Hankel functions [1, Formulas 9.2.7–9.2.10] into (4.27), we obtain

$$\Psi_{\lambda_v}(\zeta) \sim \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \left[I + \sum_{k=1}^{\infty} \frac{i^k}{2^{k+1}\zeta^k} \begin{pmatrix} (-1)^k s_{\lambda_v,k} & -it_{\lambda_v,k} \\ i(-1)^k t_{\lambda_v,k} & s_{\lambda_v,k} \end{pmatrix} \right] \\ \times e^{\frac{\pi i}{4}\sigma_3} e^{-\frac{1}{2}\lambda_v \pi i \sigma_3} e^{-i\zeta\sigma_3}, \tag{5.9}$$

as $\zeta \to \infty$, uniformly for ζ in the first quadrant. Here, the constants $s_{\lambda_v,k}$ and $t_{\lambda_v,k}$ are given by

$$s_{\lambda_v,k} = \left(\lambda_v + \frac{1}{2}, k\right) + \left(\lambda_v - \frac{1}{2}, k\right), \quad t_{\lambda_v,k} = \left(\lambda_v + \frac{1}{2}, k\right) - \left(\lambda_v - \frac{1}{2}, k\right), \tag{5.10}$$

where

$$(v, k) = \frac{(4v^2 - 1)(4v^2 - 9)\dots(4v^2 - (2k-1)^2)}{2^{2k} k!}.$$

For $z \in \partial U_{\delta, x_v} \cap K_{x_v}^{\mathrm{I}}$ we have, since f_{x_v} is a one-to-one conformal mapping on U_{δ, x_v} that $nf_{x_v}(z)$ lies in the first quadrant, see Fig. 6. So, we may use (5.9) to determine the asymptotic expansion of $\Psi_{\lambda_v}(nf_{x_v}(z))$ as $n \to \infty$. Since $\operatorname{Im} z > 0$ we have by (4.40),

$$e^{-inf_{x_v}(z)} = \varphi_+(x_v)^{-n}\varphi(z)^n.$$

Therefore, by (5.9),

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \Psi_{\lambda_v}(nf_{x_v}(z))\varphi(z)^{-n\sigma_3} \\ \sim \left[I + \sum_{k=1}^{\infty} \frac{i^k}{2^{k+1} f_{x_v}(z)^k n^k} \begin{pmatrix} (-1)^k s_{\lambda_v,k} & -it_{\lambda_v,k} \\ i(-1)^k t_{\lambda_v,k} & s_{\lambda_v,k} \end{pmatrix} \right] e^{\frac{\pi i}{4}\sigma_3} \varphi_+(x_v)^{-n\sigma_3} e^{-\frac{1}{2}\lambda_v \pi i \sigma_3},$$

as $n \to \infty$, uniformly for $z \in \partial U_{\delta, x_v} \cap K_{x_v}^{\mathrm{I}}$. Inserting this into (5.8) we have by (4.49), and the fact that $\varphi_+(x_v)^n$ remains bounded and bounded away from 0 as $n \to \infty$

(since $|\varphi_+(x_v)| = 1$),

$$\Delta(z) \sim \sum_{k=1}^{\infty} \frac{i^k}{2^{k+1} f_{x_v}(z)^k} E_v(z) \varphi_+(x_v)^{n\sigma_3} \begin{pmatrix} (-1)^k s_{\lambda_v,k} & -t_{\lambda_v,k} \\ -(-1)^k t_{\lambda_v,k} & s_{\lambda_v,k} \end{pmatrix} \times \varphi_+(x_v)^{-n\sigma_3} E_v^{-1}(z) \frac{1}{n^k}, \tag{5.11}$$

as $n \to \infty$, uniformly for $z \in \partial U_{\delta,x_v} \cap K^{\mathrm{I}}_{x_v}$. Similarly, we find the same asymptotic expansion on the other regions of $\partial U_{\delta,x_v}$. The details are left to the reader. Thus, for $z \in \partial U_{\delta,x_v}$ the coefficients of the expansion (5.6) for Δ are given by

$$\Delta_k(z,n) = \frac{i^k}{2^{k+1} f_{x_v}(z)^k} E_v(z) \varphi_+(x_v)^{n\sigma_3} \begin{pmatrix} (-1)^k s_{\lambda_v,k} & -t_{\lambda_v,k} \\ -(-1)^k t_{\lambda_v,k} & s_{\lambda_v,k} \end{pmatrix} \times \varphi_+(x_v)^{-n\sigma_3} E_v^{-1}(z). \tag{5.12}$$

Remark 5.3. These coefficients depend on n through the factors $\varphi_+(x_v)^{\pm n\sigma_3}$. Since $|\varphi_+(x_v)| = 1$, the coefficients $\Delta_k(z,n)$ remain bounded as $n \to \infty$, which is necessary to get an asymptotic expansion of form (5.6).

We note that $f_{x_v}^{-k}$ is analytic in $\mathbb{C} \setminus ((-\infty, -1] \cup [1, \infty))$ except for a pole of order k at x_v, see the discussion at the end of Section 4.2. From the proof of Proposition 4.5 and the fact that $\det E_v = 1$ we have that E_v as well as E_v^{-1} are analytic in $U \setminus ((-\infty, x_{v-1}] \cup [x_{v+1}, \infty))$. So, the restriction of $\Delta_k(\cdot, n)$ to $\partial U_{\delta,x_v}$ has a meromorphic continuation to a neighborhood U_{δ_0,x_v} of x_v for some $\delta_0 > \delta$, with a pole of order k at x_v.

5.2. *Asymptotic expansion of R*

We recall that Δ possesses an asymptotic expansion in powers of $1/n$ of form (5.6) with oscillatory terms in the expansion. Following the argument that leads to [7, (4.115)], this implies that R itself possesses an asymptotic expansion in powers of $1/n$ given by

$$R(z; n, w) \sim I + \sum_{k=1}^{\infty} \frac{R_k(z,n)}{n^k}, \quad \text{as } n \to \infty, \tag{5.13}$$

uniformly for $z \in \mathbb{C} \setminus \Sigma_R$. Here, for every k and n,

$$R_k(\cdot, n) \text{ is analytic in } \mathbb{C} \setminus \left(\bigcup_{v=0}^{p+1} \partial U_{\delta,x_v} \right), \tag{5.14}$$

and

$$R_k(z,n) = O(1/z), \quad \text{as } z \to \infty. \tag{5.15}$$

The n-dependance in the coefficients $R_k(z,n)$ arises through the oscillatory terms in the expansion of Δ.

We will now determine, similar as in [16, Section 8], the coefficient $R_1(z,n)$ explicitly. Expanding the jump relation $R_+ = R_-(I+\Delta)$, and collecting the terms with $1/n$ we have

$$R_{1,+}(s,n) - R_{1,-}(s,n) = \Delta_1(s,n), \quad \text{for } s \in \bigcup_{v=0}^{p+1} \partial U_{\delta,x_v}, \tag{5.16}$$

which is, together with (5.14) and (5.15) an additive RH problem. This can easily be solved using the Sokhotskii–Plemelj formula, but in our case we can write down an explicit solution as follows. Since $\Delta_1(z,n)$ is analytic in neighborhoods of $z=\pm 1$ and $z = x_v$ for $v = 1,\dots,p$, except for simple poles at those points, see Section 5.1, we can write

$$\Delta_1(z,n) = \frac{A^{(1)}(n)}{z-1} + O(1), \quad \text{as } z \to 1,$$
$$\Delta_1(z,n) = \frac{B^{(1)}(n)}{z+1} + O(1), \quad \text{as } z \to -1,$$

and

$$\Delta_1(z,n) = \frac{C_v^{(1)}(n)}{z-x_v} + O(1), \quad \text{as } z \to x_v,$$

for certain constant matrices $A^{(1)}(n), B^{(1)}(n)$ and $C_v^{(1)}(n)$.

Remark 5.4. Since $\Delta_1(s,n)$ is independent of n for $s \in \partial U_{\delta,-1} \cup \partial U_{\delta,1}$, see Section 5.1, the residues $A^{(1)}(n)$ and $B^{(1)}(n)$ of $\Delta_1(z,n)$ at $z=1$ and $z=-1$, respectively, are also independent of n. The n-dependance of the residue $C_v^{(1)}(n)$ at $z=x_v$ follows from the oscillatory terms $\varphi_+(x_v)^{\pm n\sigma_3}$ in $\Delta_1(s,n)$ near x_v, see (5.12).

By inspection we then see that

$$R_1(z,n) = \begin{cases} \dfrac{A^{(1)}(n)}{z-1} + \dfrac{B^{(1)}(n)}{z+1} + \displaystyle\sum_{v=1}^{p} \dfrac{C_v^{(1)}(n)}{z-x_v}, & \text{for } z \in \mathbb{C} \setminus \left(\displaystyle\bigcup_{v=0}^{p+1} U_{\delta,x_v} \right), \\ \dfrac{A^{(1)}(n)}{z-1} + \dfrac{B^{(1)}(n)}{z+1} + \displaystyle\sum_{v=1}^{p} \dfrac{C_v^{(1)}(n)}{z-x_v} - \Delta_1(z,n), & \text{for } z \in \displaystyle\bigcup_{v=0}^{p+1} U_{\delta,x_v}, \end{cases} \tag{5.17}$$

satisfies the additive RH problem (5.14)–(5.16). So, we need to determine the constant matrices $A^{(1)}(n), B^{(1)}(n)$ and $C_v^{(1)}(n)$ for $v=1,\dots,p$. For $A^{(1)}(n)$ and $B^{(1)}(n)$ we have found, see [16, Section 8],

$$A^{(1)}(n) = \frac{4\alpha^2-1}{16} D_\infty^{\sigma_3} \begin{pmatrix} -1 & i \\ i & 1 \end{pmatrix} D_\infty^{-\sigma_3}, \tag{5.18}$$

$$B^{(1)}(n) = \frac{4\beta^2 - 1}{16} D_\infty^{\sigma_3} \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix} D_\infty^{-\sigma_3}, \tag{5.19}$$

which is clearly independent of n. It remains to determine the residue $C_v^{(1)}(n)$ of $\Delta_1(z,n)$ at $z = x_v$, for every $v = 1, \dots, p$. Fix $v \in \{1, \dots, p\}$. Since $E_v(z)$ and $E_v^{-1}(z)$ are analytic in a neighborhood of $z = x_v$, and since

$$f_{x_v}(z)^{-1} = \sqrt{1 - x_v^2}\,\frac{1}{z - x_v} + O(1), \quad \text{as } z \to x_v,$$

we have by (5.10) and (5.12) that the residue $C_v^{(1)}(n)$ of $\Delta_1(z,n)$ at $z = x_v$ is given by

$$C_v^{(1)}(n) = \frac{i}{4}\sqrt{1 - x_v^2} E_v(x_v) \varphi_+(x_v)^{n\sigma_3} \begin{pmatrix} -2\lambda_v^2 & -2\lambda_v \\ 2\lambda_v & 2\lambda_v^2 \end{pmatrix} \varphi_+(x_v)^{-n\sigma_3} E_v^{-1}(x_v). \tag{5.20}$$

We want to simplify this expression. So, we need to find convenient expressions for $E_v(x_v)$ and $E_v^{-1}(x_v)$, and substitute these into (5.20). Since E_v is analytic near x_v we determine $E_v(x_v)$ by the following limit:

$$E_v(x_v) = \lim_{x \downarrow x_v;\, x \in \mathbb{R}} E_v(x) = \lim_{x \downarrow x_v;\, x \in \mathbb{R}} E_{v,+}(x).$$

Here, we take the limit from x to x_v on the real axis from the right. The last equality follows from the fact that E_v has no jumps on (x_v, x_{v+1}), see Proposition 4.5. From (3.18) and (4.49) we then find

$$E_v(x_v) = \lim_{x \downarrow x_v;\, x \in \mathbb{R}} D_\infty^{\sigma_3} \begin{pmatrix} \dfrac{a_+(x) + a_+(x)^{-1}}{2} & \dfrac{a_+(x) - a_+(x)^{-1}}{2i} \\ \dfrac{a_+(x) - a_+(x)^{-1}}{-2i} & \dfrac{a_+(x) + a_+(x)^{-1}}{2} \end{pmatrix} \times \left(\frac{W_{x_v,+}(x)}{D_+(x)} \right)^{\sigma_3} e^{\frac{1}{2}\lambda_v \pi i \sigma_3}. \tag{5.21}$$

The Szegő function satisfies $D_+(x) = \sqrt{w(x)} e^{-i\psi_v(x)}$ for $x \in (x_v, x_{v+1})$, see Lemma 3.4, where ψ_v is given by (1.15). By (4.8) we have $W_{x_v,+}(x) = \sqrt{w(x)} e^{-\lambda_v \pi i}$, so that by (1.6) and (1.15)

$$\lim_{x \downarrow x_v;\, x \in \mathbb{R}} \frac{W_{x_v,+}(x)}{D_+(x)} e^{\frac{1}{2}\lambda_v \pi i} = e^{i\psi_v(x_v)} e^{-\frac{1}{2}\lambda_v \pi i} = e^{-\frac{1}{2}\Phi_v i}.$$

Inserting this into (5.21) and using the following identities, which hold for $x \in (-1, 1)$,

$$\frac{a_+(x) + a_+(x)^{-1}}{2} = \frac{e^{-\frac{\pi i}{4}}}{\sqrt{2}(1 - x^2)^{1/4}} \varphi_+(x)^{1/2}, \tag{5.22}$$

$$\frac{a_+(x) - a_+(x)^{-1}}{2i} = \frac{e^{-\frac{\pi i}{4}}}{\sqrt{2}(1 - x^2)^{1/4}} i\varphi_+(x)^{-1/2}, \tag{5.23}$$

we then find

$$E_v(x_v) = \frac{e^{-\frac{\pi i}{4}}}{\sqrt{2}(1-x_v^2)^{1/4}} D_\infty^{\sigma_3} \begin{pmatrix} \varphi_+(x_v)^{1/2} & i\varphi_+(x_v)^{-1/2} \\ -i\varphi_+(x_v)^{-1/2} & \varphi_+(x_v)^{1/2} \end{pmatrix} e^{-\frac{1}{2}\Phi_v i\sigma_3}. \tag{5.24}$$

Taking inverse, we find

$$E_v^{-1}(x_v) = \frac{e^{-\frac{\pi i}{4}}}{\sqrt{2}(1-x_v^2)^{1/4}} e^{\frac{1}{2}\Phi_v i\sigma_3} \begin{pmatrix} \varphi_+(x_v)^{1/2} & -i\varphi_+(x_v)^{-1/2} \\ i\varphi_+(x_v)^{-1/2} & \varphi_+(x_v)^{1/2} \end{pmatrix} D_\infty^{-\sigma_3}. \tag{5.25}$$

Now, we insert (5.24) and (5.25) into (5.20). Using the identity $\varphi_+(x_v) = \exp(i\arccos x_v)$ we then find after a straightforward calculation that the residue $C_v^{(1)}(n)$ of $\Delta_1(z,n)$ at $z = x_v$ is given by

$$C_v^{(1)}(n) = D_\infty^{\sigma_3} \begin{pmatrix} C_{v,11}(n) & C_{v,12}(n) \\ C_{v,21}(n) & -C_{v,11}(n) \end{pmatrix} D_\infty^{-\sigma_3}, \tag{5.26}$$

where

$$C_{v,11}(n) = -\frac{1}{2}\lambda_v^2 x_v + \frac{1}{2}\lambda_v \sin(2n\arccos x_v - \Phi_v), \tag{5.27}$$

$$\begin{aligned} C_{v,12}(n) = &\frac{i}{2}\lambda_v^2 - \frac{i}{2}\lambda_v x_v \sin(2n\arccos x_v - \Phi_v) \\ &- \frac{i}{2}\lambda_v\sqrt{1-x_v^2}\cos(2n\arccos x_v - \Phi_v), \end{aligned} \tag{5.28}$$

$$\begin{aligned} C_{v,21}(n) = &\frac{i}{2}\lambda_v^2 - \frac{i}{2}\lambda_v x_v \sin(2n\arccos x_v - \Phi_v) \\ &+ \frac{i}{2}\lambda_v\sqrt{1-x_v^2}\cos(2n\arccos x_v - \Phi_v). \end{aligned} \tag{5.29}$$

This ends the determination of $R_1(z,n)$.

For general k, we get that $R_k(z,n)$ in the region $\mathbb{C}\backslash(\bigcup_{v=0}^{p+1} U_{\delta,x_v})$ is a rational function with poles at ± 1 and at the algebraic singularities x_v. The residues at 1 and -1 are denoted by $A^{(k)}(n)$ and $B^{(k)}(n)$ respectively and may depend on n. The residue at every x_v depends on n and is denoted by $C_v^{(k)}(n)$. We then get

$$R_k(z,n) = \frac{A^{(k)}(n)}{z-1} + \frac{B^{(k)}(n)}{z+1} + \sum_{v=1}^{p} \frac{C_v^{(k)}(n)}{z-x_v} + O(1/z^2), \quad \text{as } z\to\infty.$$

The residues $A^{(k)}(n), B^{(k)}(n)$ and $C_v^{(k)}(n)$ can be determined in a similar fashion, but for our purpose it suffices to know $R_1(z,n)$.

5.3. *Proof of Theorem 1.1*

We are now ready to determine a complete asymptotic expansion of the recurrence coefficients a_n and b_n. The idea is to insert the asymptotic expansion (5.13) of R into (5.1) and (5.2).

Proof of Theorem 1.1. We recall that, see (5.1),

$$a_n^2 = \lim_{z\to\infty}\left(-\frac{D_\infty^2}{2i} + zR_{12}(z;n,w)\right)\left(zR_{21}(z;n,w) + \frac{1}{2iD_\infty^2}\right).$$

We may take the limit $z\to\infty$ in the asymptotic expansion (5.13) of R, cf. [16, Section 9], to obtain

$$a_n^2 \sim \left(-\frac{D_\infty^2}{2i} + \sum_{k=1}^{\infty}\frac{A_{12}^{(k)}(n) + B_{12}^{(k)}(n) + \sum_{v=1}^{p} C_{v,12}^{(k)}(n)}{n^k}\right)$$
$$\times\left(\sum_{k=1}^{\infty}\frac{A_{21}^{(k)}(n) + B_{21}^{(k)}(n) + \sum_{v=1}^{p} C_{v,21}^{(k)}(n)}{n^k} + \frac{1}{2iD_\infty^2}\right), \tag{5.30}$$

as $n\to\infty$. Expanding this we find a complete asymptotic expansion of a_n^2, and this leads to a complete asymptotic expansion of a_n in powers of $1/n$ as $n\to\infty$. By (5.18), (5.19), (5.26), (5.28)–(5.30) the first terms in the asymptotic expansion of a_n^2 are

$$a_n^2 = \frac{1}{4} + \frac{1}{2i}\left[D_\infty^{-2}\left(A_{12}^{(1)}(n) + B_{12}^{(1)}(n) + \sum_{v=1}^{p} C_{v,12}^{(1)}(n)\right)\right.$$
$$\left. -\ D_\infty^2\left(A_{21}^{(1)}(n) + B_{21}^{(1)}(n) + \sum_{v=1}^{p} C_{v,21}^{(1)}(n)\right)\right]\frac{1}{n} + O\left(\frac{1}{n^2}\right)$$
$$= \frac{1}{4} - \sum_{v=1}^{p}\frac{\lambda_v}{2}\sqrt{1-x_v^2}\cos\left(2n\arccos x_v - \Phi_v\right)\frac{1}{n} + O\left(\frac{1}{n^2}\right), \tag{5.31}$$

as $n\to\infty$. From this we then get, after a simple calculation, that the coefficient with the $1/n$ term in the asymptotic expansion of a_n is given by (1.4). So, the statements about the recurrence coefficient a_n are proved.

Similarly, we can prove the statements about the recurrence coefficient b_n. If we take in (5.2) the limit $z\to\infty$ in the asymptotic expansion (5.13) of R, cf. [16, Section 9], we find

$$b_n \sim \lim_{z\to\infty} -z\left(\sum_{k=1}^{\infty}\frac{(R_k)_{11}(z,n+1)}{(n+1)^k} + \sum_{k=1}^{\infty}\frac{(R_k)_{22}(z,n)}{n^k}\right)$$
$$= -\sum_{k=1}^{\infty}\left(\frac{A_{11}^{(k)}(n+1) + B_{11}^{(k)}(n+1) + \sum_{v=1}^{p} C_{v,11}^{(k)}(n+1)}{(n+1)^k}\right.$$
$$\left. + \frac{A_{22}^{(k)}(n) + B_{22}^{(k)}(n) + \sum_{v=1}^{p} C_{v,22}^{(k)}(n)}{n^k}\right), \tag{5.32}$$

as $n\to\infty$. From this we get a complete asymptotic expansion of b_n in powers of $1/n$, and by (5.18), (5.19), (5.26), (5.27) and (5.32) the coefficient with the $1/n$ term in the asymptotic expansion of b_n is given by

$$\begin{aligned} B_1(n) = & -(A_{11}^{(1)}(n+1) + A_{22}^{(1)}(n)) - (B_{11}^{(1)}(n+1) + B_{22}^{(1)}(n)) \\ & - \sum_{v=1}^{p} (C_{v,11}^{(1)}(n+1) + C_{v,22}^{(1)}(n)) \\ = & - \sum_{v=1}^{p} \frac{\lambda_v}{2} [\sin((2n+1)\arccos x_v - \Phi_v + \arccos x_v) \\ & - \sin((2n+1)\arccos x_v - \Phi_v - \arccos x_v)] \\ = & - \sum_{v=1}^{p} \lambda_v \sqrt{1 - x_v^2} \cos((2n+1)\arccos x_v - \Phi_v). \end{aligned} \tag{5.33}$$

Therefore, the theorem is proven. □

6. Asymptotics of the orthonormal polynomials

In this section, we determine the asymptotic behavior of the orthonormal polynomials p_n, as $n \to \infty$, near the algebraic singularities. This will be done by going back in the series of transformations $Y \mapsto T \mapsto S \mapsto R$.

Since $p_n = \gamma_n \pi_n$, we first want to know the asymptotic behavior of the leading coefficient γ_n as $n \to \infty$. Similar considerations as in [16, proof of Theorem 1.6] show that

$$\gamma_n = \frac{2^n}{\sqrt{\pi}\, D_\infty}(1 + O(1/n)), \quad \text{as } n \to \infty, \tag{6.1}$$

with an error term that has a full asymptotic expansion in powers of $1/n$.

Now, we are ready to prove Theorem 1.3.

Proof of Theorem 1.3. Let z be in the right upper part of the lens inside the disk U_{δ,x_v} around x_v. From (3.1), (3.6), (3.25) and (4.42) we establish that

$$\begin{aligned} Y(z) = & 2^{-n\sigma_3} R(z) E_{n,x_v}(z) \Psi_{\lambda_v}(n f_{x_v}(z)) W_{x_v}(z)^{-\sigma_3} \\ & \times \varphi(z)^{-n\sigma_n} \begin{pmatrix} 1 & 0 \\ w(z)^{-1}\varphi(z)^{-2n} & 1 \end{pmatrix} \varphi(z)^{n\sigma_3}. \end{aligned} \tag{6.2}$$

Since z is in the right upper part of the lens inside the disk U_{δ,x_v} we have $W_{x_v}(z) = w(z)^{1/2} e^{-\lambda_v \pi i}$, see (4.8). Inserting this into (6.2) we then obtain that the first column of Y is given by

$$\begin{pmatrix} Y_{11}(z) \\ Y_{21}(z) \end{pmatrix} = w(z)^{-1/2} 2^{-n\sigma_3} R(z) E_{n,x_v}(z) \Psi_{\lambda_v}(n f_{x_v}(z)) e^{\lambda_v \pi i \sigma_3} \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{6.3}$$

For our choice of z we have that $0 < \arg nf_{x_v}(z) < \pi/4$, see Fig. 6. So, we have to use (4.26) to evaluate $\Psi_{\lambda_v}(nf_{x_v}(z))$. From [1, Formulas 9.1.3 and 9.1.4], which connect the Hankel functions with the usual J-Bessel functions, we then have

$$\Psi_{\lambda_v}(nf_{x_v}(z))e^{\lambda_v \pi i \sigma_3}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = e^{-\frac{\pi i}{4}}\sqrt{\pi}(nf_{x_v}(z))^{1/2}\begin{pmatrix} J_{\lambda_v+\frac{1}{2}}(nf_{x_v}(z)) \\ J_{\lambda_v-\frac{1}{2}}(nf_{x_v}(z)) \end{pmatrix}.$$

Inserting this into (6.3), taking the limit $z \to x \in (x_v, x_v + \delta)$, and noting that $f_{x_v,+}(x) = \arccos x_v - \arccos x = \theta_v$, which follows from (4.40), we obtain

$$\begin{pmatrix} Y_{11}(x) \\ Y_{21}(x) \end{pmatrix} = \frac{e^{-\frac{\pi i}{4}}\sqrt{\pi}}{\sqrt{w(x)}}(n\theta_v)^{1/2}2^{-n\sigma_3}R(x)E_{n,x_v,+}(x)\begin{pmatrix} J_{\lambda_v+\frac{1}{2}}(n\theta_v) \\ J_{\lambda_v-\frac{1}{2}}(n\theta_v) \end{pmatrix}. \tag{6.4}$$

Now, we want to determine a convenient expression for $E_{n,x_v,+}(x)$. We have to use (4.48) and (4.49) to evaluate $E_{n,x_v,+}(x)$. By (3.18), (5.22) and (5.23), and from the fact that

$$\frac{W_{x_v,+}(x)}{D_+(x)} = \exp(i(\psi_v(x) - \lambda_v\pi)),$$

see (3.20) and (4.8), we then obtain

$$\begin{aligned} E_{n,x_v,+}(x) = &\frac{e^{-\frac{\pi i}{4}}}{\sqrt{2}(1-x^2)^{1/4}}D_\infty^{\sigma_3}\begin{pmatrix} \varphi_+(x)^{1/2} & i\varphi_+(x)^{-1/2} \\ -i\varphi_+(x)^{-1/2} & \varphi_+(x)^{1/2} \end{pmatrix} \\ &\times \left(\frac{W_{x_v,+}(x)}{D_+(x)}\right)^{\sigma_3} e^{\frac{1}{2}\lambda_v \pi i \sigma_3}\varphi_+(x_v)^{n\sigma_3}e^{-\frac{\pi i}{4}\sigma_3}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \\ = &\frac{e^{-\frac{\pi i}{4}}}{2(1-x^2)^{1/4}}D_\infty^{\sigma_3}\begin{pmatrix} \varphi_+(x)^{1/2} & i\varphi_+(x)^{-1/2} \\ -i\varphi_+(x)^{-1/2} & \varphi_+(x)^{1/2} \end{pmatrix} \\ &\times e^{i(\psi_v(x)-\frac{1}{2}\lambda_v\pi + n\arccos x_v - \frac{\pi}{4})\sigma_3}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}. \end{aligned}$$

The latter expression can be written as

$$\begin{aligned} E_{n,x_v,+}(x) = &\frac{e^{-\frac{\pi i}{4}}}{2(1-x^2)^{1/4}}D_\infty^{\sigma_3}\begin{pmatrix} e^{i\zeta_1(x)} & ie^{-i\zeta_1(x)} \\ -ie^{i\zeta_2(x)} & e^{-i\zeta_2(x)} \end{pmatrix}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \\ = &\frac{e^{-\frac{\pi i}{4}}}{(1-x^2)^{1/4}}D_\infty^{\sigma_3}\begin{pmatrix} i\sin\zeta_1(x) & i\cos\zeta_1(x) \\ \sin\zeta_2(x) & \cos\zeta_2(x) \end{pmatrix}, \end{aligned} \tag{6.5}$$

where ζ_1 and ζ_2 are given by (1.14). Inserting this into (6.4) and using the fact that $Y_{11} = \pi_n$, we obtain

$$\pi_n(x) = \frac{\sqrt{\pi}D_\infty}{2^n}\frac{(n\theta_v)^{1/2}}{\sqrt{w(x)}(1-x^2)^{1/4}} \times \left[R_{11}(x)(\cos\zeta_1(x)J_{\lambda_v-\frac{1}{2}}(n\theta_v) + \sin\zeta_1(x)J_{\lambda_v+\frac{1}{2}}(n\theta_v)) - \frac{i}{D_\infty^2}R_{12}(x)(\cos\zeta_2(x)J_{\lambda_v-\frac{1}{2}}(n\theta_v) + \sin\zeta_2(x)J_{\lambda_v+\frac{1}{2}}(n\theta_v))\right].$$

Since $p_n = \gamma_n\pi_n$, from (6.1), and from the facts that $R_{11}(x) = 1 + O(1/n)$ and $R_{12}(x) = O(1/n)$ as $n \to \infty$, with error terms that hold uniformly for $x \in (x_v, x_v + \delta)$ and that have a full asymptotic expansion in powers of $1/n$, the theorem is then proven. □

Acknowledgments

I thank my advisor Arno Kuijlaars for useful discussions, good ideas, careful reading, and his great support.

References

[1] M. Abramowitz, I.A. Stegun, Handbook of Mathematical Functions, Dover Publications, New York, 1968.
[2] V.M. Badkov, Convergence in the mean and almost everywhere of Fourier series in polynomials orthogonal on an interval, Math. USSR, Sbornik 24 (1974) 223–256.
[3] P. Bleher, A. Its, Semiclassical asymptotics of orthogonal polynomials, Riemann–Hilbert problem, and universality in the matrix model, Ann. of Math. (2) 150 (1) (1999) 185–266.
[4] T.S. Chihara, An Introduction to Orthogonal Polynomials, Gordon and Breach, New York, 1978.
[5] P. Deift, Orthogonal Polynomials and Random Matrices: A Riemann–Hilbert Approach, Courant Lecture Notes in Mathematics Vol. 3, Courant Institute of Mathematical Sciences, New York University, 1999.
[6] P. Deift, T. Kriecherbauer, K.T.-R. McLaughlin, S. Venakides, X. Zhou, Uniform asymptotics for polynomials orthogonal with respect to varying exponential weights and applications to universality questions in random matrix theory, Commun. Pure Appl. Math. 52 (11) (1999) 1335–1425.
[7] P. Deift, T. Kriecherbauer, K.T.-R. McLaughlin, S. Venakides, X. Zhou, Strong asymptotics of orthogonal polynomials with respect to exponential weights, Commun. Pure Appl. Math. 52 (12) (1999) 1491–1552.
[8] P. Deift, T. Kriecherbauer, K.T.-R. McLaughlin, S. Venakides, X. Zhou, A Riemann–Hilbert approach to asymptotic questions for orthogonal polynomials, J. Comput. Appl. Math. 133 (1–2) (2001) 47–63.
[9] P. Deift, X. Zhou, A steepest descent method for oscillatory Riemann–Hilbert problems, Asymptotics for the MKdV equation, Ann. of Math. (2) 137 (2) (1993) 295–368.
[10] T. Erdélyi, P. Nevai, A.P. Magnus, Generalized Jacobi weights, Christoffel functions, and Jacobi polynomials, SIAM J. Math. Anal. 25 (2) (1994) 602–614.

[11] A.S. Fokas, A.R. Its, A.V. Kitaev, The isomonodromy approach to matrix models in 2D quantum gravity, Commun. Math. Phys. 147 (2) (1992) 395–430.
[12] F.D. Gakhov, Boundary Value Problems, 2nd Edition, Dover Publications, New York, 1990.
[13] L.B. Golinskii, Reflection coefficients for the generalized Jacobi weight functions, J. Approx. Theory 87 (1) (1994) 117–126.
[14] T. Kriecherbauer, K.T.-R. McLaughlin, Strong asymptotics of polynomials orthogonal with respect to Freud weights, Internat. Math. Res. Not. 1999 (6) (1999) 299–333.
[15] A.B.J. Kuijlaars, K.T.-R. McLaughlin, Riemann–Hilbert analysis for Laguerre polynomials with large negative parameter, Comput. Methods Function Theory 1 (1) (2001) 205–233.
[16] A.B.J. Kuijlaars, K.T.-R. McLaughlin, W. Van Assche, M. Vanlessen, The Riemann–Hilbert approach to strong asymptotics for orthogonal polynomials, Preprint, 2001, http://xxx.lanl.gov/abs/math.CA/0111252, to appear in Adv. Math.
[17] A.P. Magnus, Asymptotics for the simplest generalized Jacobi polynomials recurrence coefficients from Freud's equation: numerical explorations, Ann. Numer. Math. 2 (1–4) (1995) 311–325.
[18] P.G. Nevai, Orthogonal Polynomials, Memoirs of the American Mathematical Society, Providence, RI, 1979.
[19] G. Szegő, Orthogonal Polynomials, 4th Edition, American Mathematical Society, Providence, RI, 1975.
[20] P. Vértesi, Asymptotics of derivatives of orthogonal polynomials based on generalized Jacobi weights, Some new theorems and applications, Internat. Ser. Numer. Math. 132 (1999) 329–339.
[21] P. Vértesi, Uniform asymptotics of derivatives of orthogonal polynomials based on generalized Jacobi weights, Acta Math. Hungar. 85 (1–2) (1999) 97–130.